Chapter 10: Multiple Regression Analysis – Introduction

Chapter 10 Outline

- Simple versus Multiple Regression Analysis
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- A One-Tailed Test: Downward Sloping Demand Theory
- A Two-Tailed Test: No Money Illusion Theory
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  - Constant Elasticity Demand Model and the No Money Illusion Theory
  - Calculating Prob[Results IF H₀ True]: Clever Algebraic Manipulation
    - Cleverly Define a New Coefficient That Equals 0 When H₀ Is True
    - Reformulate the Model to Incorporate the New Coefficient
    - Estimate the Parameters of the New Model
    - Use the Tails Probability to Calculate Prob[Results IF H₀ True]

Chapter 10 Prep Questions

1. Consider the following constant elasticity model:

\[ Q = \beta_{\text{Const}} P^{\beta_p} I^{\beta_i} \text{ChickP}^{\beta_{\text{Chick}}} \]

where

- \( Q \) = Quantity of beef demanded
- \( P \) = Price of beef (the good’s own price)
- \( I \) = Household Income
- \( \text{ChickP} \) = Price of chicken

a. Show that if \( \beta_{\text{CP}} = -\beta_p - \beta_i \), then

\[ Q = \beta_{\text{Const}} \left( \frac{P}{\text{ChickP}} \right)^{\beta_p} \left( \frac{I}{\text{ChickP}} \right)^{\beta_i} \]

b. If \( \beta_{\text{CP}} = -\beta_p - \beta_i \), what happens to the quantity of beef demanded when the price of beef (the good’s own price, \( P \)), income (\( I \)), and the price of chicken (\( \text{ChickP} \)) all double?

c. If \( \beta_p + \beta_i + \beta_{\text{CP}} = 0 \), what happens to the quantity of beef demanded when the price of beef (the good’s own price, \( P \)), income (\( I \)), and the price of chicken (\( \text{ChickP} \)) all double?

2. Again, consider the following constant elasticity model:

\[ Q = \beta_{\text{Const}} P^{\beta_p} I^{\beta_i} \text{ChickP}^{\beta_{\text{Chick}}} \]
What does \( \log(Q) \) equal, where \( \log \) is the natural logarithm?

3. Consider the following model:

\[
\log(Q) = \log(\beta_{\text{Const}}) + \beta_P \log(P) + \beta_I \log(I) + \beta_{\text{CP}} \log(\text{ChickP})
\]

Let \( \beta_{\text{Clever}} = \beta_P + \beta_I + \beta_{\text{CP}} \). Show that

\[
\log(Q) = \log(\beta_{\text{Const}}) + \beta_P \{ \log(P) - \log(\text{ChickP}) \} + \beta_I \{ \log(I) - \log(\text{ChickP}) \} + \beta_{\text{Clever}} \log(\text{ChickP})
\]

**Simple versus Multiple Regression Analysis**

Thus far, we have focused our attention on **simple regression analysis** in which the model assumes that only a single explanatory variable affects the dependent variable. In the real world, however, a dependent variable typically depends on many explanatory variables. For example, while economic theory teaches that the quantity of a good demanded depends on the good’s own price, theory also tells us that the quantity depends on other factors also: income, the price of other goods, etc. **Multiple regression analysis** allows us to assess such theories.

**Goal of Multiple Regression Analysis**

- Multiple regression analysis attempts to sort out the individual effect of each explanatory variable.
- An explanatory variable’s coefficient estimate allows us to estimate the change in the dependent variable resulting from a change in that particular explanatory variable while all other explanatory variables remain constant.
A One-Tailed Test: Downward Sloping Demand Theory
We begin by explicitly stating the theory:

**Downward Sloping Demand Theory:** The quantity of a good demanded by a household depends on its price and other relevant factors. When the good’s own price increases while all other relevant factors remain constant, the quantity demanded decreases.

**Project:** Assess the downward sloping demand theory.

Graphically, the theory is illustrated by a downward sloping demand curve.

![Downward Sloping Demand Curve](image)

Figure 10.1: *Downward Sloping Demand Curve*

When we draw a demand curve for a good, we implicitly assume that all factors relevant to demand other than that good’s own price remain the constant.

We shall focus on the demand for a particular good, beef, to illustrate the importance of multiple regression analysis. We now apply the hypothesis testing steps.

**Step 0:** Formulate a model reflecting the theory to be tested.

We use with a linear demand model to test the theory. Naturally, the quantity of beef demanded depends on its own price, the price of beef itself. Furthermore, we postulate that the quantity of beef demanded also depends on income and the price of chicken. In other words, our model proposes that the factors relevant to the demand for beef, other than beef’s own price, are income and the price of chicken.

\[
Q = \beta_{\text{Const}} + \beta_P P_t + \beta_I I_t + \beta_{\text{ChickP}} \text{Chick}P_t + e_t
\]

where

- \(Q_t\) = Quantity of beef demanded
- \(P_t\) = Price of beef (the good's own price)
- \(I_t\) = Household Income
- \(\text{Chick}P_t\) = Price of chicken
The theory suggests that when income and the price of chicken remain constant, an increase in the price of beef (the good’s own price) decreases the quantity of beef demanded; similarly, when income and the price chicken remain constant, a decrease in the price of beef (the good’s own price) increases the quantity of beef demanded:

When income \((I)\) and the price of chicken \((\text{ChickP})\) remain constant:

- An increase in the price of beef, the good’s own price \((P)\)
- Quantity of beef demanded \((Q)\) decreases

- A decrease in the price of beef, the good’s own price \((P)\)
- Quantity of beef demanded \((Q)\) increases

Figure 10.2: Downward Sloping Demand Curve for Beef

The theory suggests that the model’s price coefficient, \(\beta_P\), is negative:

\[
Q = \beta_{\text{Const}} + \beta_P P + \beta_I I + \beta_{\text{ChickP}} \text{ChickP} + e_i
\]

- \(P\) increases \(\downarrow\) \(Q\) decreases
- \(I\) constant \(\downarrow\) \(Q\) constant
- \(\text{ChickP}\) constant \(\downarrow\) \(Q\) constant

\(\beta_P < 0\)
Economic theory teaches that the sign of coefficients for the explanatory variables other than the good’s own price may be positive or negative. Their signs depend on the particular good in question:

- The sign of $\beta_I$ depends on whether beef is a normal or inferior good. Beef is generally regarded as a normal good; consequently, we would expect $\beta_I$ to be positive: an increase in income results in an increase in the quantity of beef demanded.
  
  Beef a normal good
  
  $\downarrow$
  
  $\beta_I > 0$

- The sign of $\beta_{CP}$ depends on whether beef and chicken are substitutes or complements. Beef and chicken are generally believed to be substitutes; consequently, we would expect $\beta_{CP}$ to be positive. An increase in the price of chicken would cause consumers to substitute beef for the now more expensive chicken; that is, an increase in the price of chicken results in an increase in the quantity of beef demanded.
  
  Beef and chicken substitutes
  
  $\downarrow$
  
  $\beta_{CP} > 0$
Step 1: Collect data, run the regression, and interpret the estimates.

**Beef Consumption Data:** Monthly time series data of beef consumption, beef prices, income, and chicken prices from 1985 and 1986.

- $Q_t$: Quantity of beef demanded in month $t$ (millions of pounds)
- $P_t$: Price of beef in month $t$ (cents per pound)
- $I_t$: Disposable income in month $t$ (billions of chained 1985 dollars)
- $\text{Chick} P_t$: Price of chicken in month $t$ (cents per pound)

**Table 10.1: Monthly Beef Demand Data from 1985 and 1986**

These data can be accessed at the following link:

[Link to MIT-BeefDemand-1985-1986.wf1 goes here.]
We now use the ordinary least squares (OLS) estimation procedure to estimate the model’s parameters:

Ordinary Least Squares (OLS)

<table>
<thead>
<tr>
<th>Dependent Variable: Q</th>
<th>Estimate</th>
<th>SE</th>
<th>t-Statistic</th>
<th>Prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>-549.4847</td>
<td>130.2611</td>
<td>-4.218333</td>
<td>0.0004</td>
</tr>
<tr>
<td>I</td>
<td>24.24854</td>
<td>11.27214</td>
<td>2.151192</td>
<td>0.0439</td>
</tr>
<tr>
<td>ChickP</td>
<td>287.3737</td>
<td>193.3540</td>
<td>1.486257</td>
<td>0.1528</td>
</tr>
<tr>
<td>Const</td>
<td>159032.4</td>
<td>61472.68</td>
<td>2.587041</td>
<td>0.0176</td>
</tr>
</tbody>
</table>

Number of Observations 24

Estimated Equation: \( \hat{Q} = 159,032 - 549.5P + 24.25I + 287.4\text{ChickP} \)

Table 10.2: Beef Demand Regression Results – Linear Model

To interpret these estimates, let us for the moment replace the numerical value of each estimate with the italicized lower case Roman letter \( b \), that we use to denote the estimate. That is, replace the estimated

- constant, 159,032, with \( b_{\text{const}} \)
- price coefficient, \(-549.5\), with \( b_p \)
- income coefficient, 24.25, with \( b_I \)
- chicken price coefficient, 287.4, with \( b_{\text{ChickP}} \).

\[ \hat{Q} = b_{\text{const}} + b_pP + b_I + b_{\text{ChickP}} \]

The coefficient estimates attempt to separate out the individual effect that each explanatory variable has on the dependent variable. To justify this, focus on the estimate of the beef price coefficient, \( b_p \). It estimates by how much the quantity of beef changes when the price of beef (the good’s own price) changes while income and the price of chicken (all other explanatory variables) remain constant. More formally, when all other explanatory variables remain constant:

\[ \Delta Q = b_p \Delta P \quad \text{or} \quad b_p = \frac{\Delta Q}{\Delta P} \]

where \( \Delta Q = \text{Change in the quantity of beef demanded} \)

\( \Delta P = \text{Change in the price of beef, the good’s own price} \)
A little algebra explains why. We begin with the equation estimating our model:
\[
\text{Est}Q = b_{\text{Const}} + b_P P + b_I + b_{CP} \text{ChickP}
\]

Now, increase the price of beef (the good’s own price) by \( \Delta P \) while keeping all other explanatory variables constant. \( \Delta Q \) estimates the resulting change in quantity of beef demanded.

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>While all other explanatory variables remain constant; that is, while ( I ) and ( \text{ChickP} ) remain constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price:</td>
<td>( P )</td>
<td>( P + \Delta P )</td>
</tr>
<tr>
<td>Quantity:</td>
<td>( \text{Est}Q )</td>
<td>( \text{Est}Q + \Delta Q )</td>
</tr>
</tbody>
</table>

In the equation estimating our model, substitute
- \( \text{Est}Q + \Delta Q \) for \( \text{Est}Q \)
  and
- \( P + \Delta P \) for \( P \):

\[
\begin{align*}
\text{Est}Q + \Delta Q &= b_{\text{Const}} + b_P (P + \Delta P) + b_I + b_{CP} \text{ChickP} \\
\Rightarrow \Delta Q &= b_P \Delta P + 0 + 0 + 0 \\
\frac{\Delta Q}{\Delta P} &= b_P 
\end{align*}
\]

while all other explanatory variables constant

To summarize:
\[
\Delta Q = b_P \Delta P \quad \text{or} \quad b_P = \frac{\Delta Q}{\Delta P} \quad \text{while all other explanatory variables (} I \text{ and} \ \text{ChickP}) \text{ remain constant}
\]

Note that the sign of \( b_P \) determines whether or not the data supports the downward sloping demand theory. A demand curve illustrates what happens to the quantity demanded when the price of the good changes while all other factors that affect demand (in the case income and the price of chicken) remain constant. \( b_P \) is the estimated “slope” of the demand curve.
The word slope has been placed in quotes. Why is this? Slope is defined as rise divided by run. As Figure 10.3 illustrates, \( b_p \) equals run over rise, however. This occurred largely by an historical accident. When economists, Alfred Marshall in particular, first developed demand and supply curves, they placed the price on the vertical axis and the quantity on the horizontal axis. Consequently, \( b_p \) actually equals the reciprocal of the estimated slope. To avoid using the awkward phrase “the reciprocal of the estimated slope” repeatedly, we place the word slope within double quotes to denote this.

Now, let us interpret the other coefficients. Using similar logic:

\[
\Delta Q = b_I \Delta I \quad \text{or} \quad b_I = \frac{\Delta Q}{\Delta I} \quad \text{while all other explanatory variables (} P \text{ and } ChickP \text{) remain constant}
\]

\( b_I \) estimates the change in quantity when income changes while all other explanatory variables (the price of beef and the price of chicken) remain constant.

\[
\Delta Q = b_{CP} \Delta ChickP \quad \text{or} \quad b_{CP} = \frac{\Delta Q}{\Delta PChick} \quad \text{while all other explanatory variables (} P \text{ and } I \text{) remain constant}
\]

\( b_{CP} \) estimates the change in quantity when the price of chicken changes while all other explanatory variables (the price of beef and income) remain constant.
What happens when the price of beef (the good’s own price), income, and the price of chicken change simultaneously? The total estimated change in the quantity of beef demanded just equals the sum of the individual changes; that is, the total estimated change in the quantity of beef demanded equals the change resulting from the change in

- the price of beef (the good’s own price)

plus

- income

plus

- the price of chicken.

The following equation expresses this succinctly:

\[
\Delta Q = b_p \Delta P + b_I \Delta I + b_{ChickP} \Delta ChickP
\]

Each term estimates the change in the dependent variable, quantity of beef demanded, resulting from a change in each individual explanatory variable.

The estimates achieve the goal:

**Goal of Multiple Regression Analysis:** Multiple regression analysis attempts to sort out the individual effect of each explanatory variable. An explanatory variable’s coefficient estimate allows us to estimate the change in the dependent variable resulting from a change in that particular explanatory variable while all other explanatory variables remain constant.

Now, let us interpret the numerical values of the coefficient estimates:

Estimated effect of a change in the price of beef (the good’s own price):

\[
\Delta Q = b_p \Delta P = -549.5 \Delta P
\]

while all other explanatory variables remain constant

**Interpretation:** The ordinary least squares (OLS) estimate of the price coefficient equals \(-549.5\); that is, we estimate that if the price of beef increases by 1 cent while income and the price of chicken remain unchanged, the quantity of beef demanded decreases by about 549.5 million pounds.
Estimated effect of a change in income:
\[ \Delta Q = b_I \Delta I = 24.25 \Delta I \] while all other explanatory variables remain constant

**Interpretation:** The ordinary least squares (OLS) estimate of the income coefficient equals 24.25; that is, we estimate that if disposable income increases by 1 billion dollars while the price of beef and the price of chicken remain unchanged, the quantity of beef demanded increases by about 24.25 million pounds.

Estimated effect of a change in the price of chicken:
\[ \Delta Q = b_{CP} \Delta ChickP = 287.4 \Delta ChickP \] while all other explanatory variables remain constant

**Interpretation:** The ordinary least squares (OLS) estimate of the chicken price coefficient equals 287.4; that is, we estimate that if the price of chicken increases by 1 cent while the price of beef and income remain unchanged, the quantity of beef demanded increases by about 287.4 million pounds.

Putting the three estimates together:
\[ \Delta Q = b_P \Delta P + b_I \Delta I + b_{CP} \Delta ChickP \]

or
\[ \Delta Q = -549.5 \Delta P + 24.25 \Delta I + 287.4 \Delta ChickP \]

We estimate that the total change in the quantity of beef demanded equals −549.5 times the change in the price of beef (the good’s own price) plus 24.25 times the change in disposable income plus 287.4 times the change in the price of chicken.

Recall that the sign of the estimate for the good’s own price coefficient, \( b_P \), determines whether or not the data support the downward sloping demand theory. \( b_P \) estimates the change in the quantity of beef demanded when the price of beef (the good’s own price) changes while the other explanatory variables, income and the price of chicken, remain constant. The theory postulates that an increase in the good’s own price decreases the quantity of beef demanded. The negative price coefficient estimate lends support to the theory.

**Critical Result:** The own price coefficient estimate is −549.5. The negative sign of the coefficient estimate suggests that an increase in the price decreases the quantity of beef demanded. This evidence supports the downward sloping theory.

Now, let us continue with the hypothesis testing steps.
Step 2: Play the cynic and challenge the results; construct the null and alternative hypotheses.

The cynic is skeptical of the evidence supporting the view that the actual price coefficient, $\beta_p$, is negative; that is, the cynic challenges the evidence and hence the downward sloping demand theory:

**Cynic’s view:** Sure, the price coefficient estimate from the regression suggests that the demand curve is downward sloping, but this is just “the luck of the draw.” In fact, the actual price coefficient, $\beta_p$, equals $0$.

- $H_0$: $\beta_p = 0$  Cynic is correct: The price of beef (the good’s own price) has no effect on quantity of beef demanded.
- $H_1$: $\beta_p < 0$  Cynic is incorrect: An increase in the price decreases quantity of beef demanded.

The null hypothesis, like the cynic, challenges the evidence: an increase in the price of beef has no effect on the quantity of beef demanded. The alternative hypothesis is consistent with the evidence: an increase in the price decreases the quantity of beef demanded.

Step 3: Formulate the question to assess the cynic’s view and the null hypothesis.

![Figure 10.4: Probability Distribution of Coefficient Estimate for the Beef Price](image)

- **Generic Question:** What is the probability that the results would be like those we obtained (or even stronger), if the cynic is correct and the price of beef actually has no impact?
- **Specific Question:** What is the probability that the coefficient estimate, $b_p$, in one regression would be $-549.5$ or less, if $H_0$ were true (if the actual price coefficient, $\beta_p$, equals $0$)?

**Answer:** Prob[Results IF Cynic Correct] or Prob[Results IF $H_0$ True].
**Step 4:** Use the general properties of the estimation procedure, the probability distribution of the estimate, to calculate Prob[Results IF \( H_0 \) True].

<table>
<thead>
<tr>
<th>OLS estimation procedure unbiased</th>
<th>If ( H_0 ) true</th>
<th>Standard error</th>
<th>Number of observations</th>
<th>Number of parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean[( b_P )] = ( \beta_P = 0 )</td>
<td>( SE[ b_P ] = 130.3 )</td>
<td>DF = 24 − 4 = 20</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We can now calculate Prob[Results IF \( H_0 \) True]. The easiest way is to use the regression results. Recall that the tails probability is reported in the Prob column. The tails probability is .0004; therefore, to calculate Prob[Results IF \( H_0 \) True] we need only divide .0004 by 2:

**Ordinary Least Squares (OLS)**

<table>
<thead>
<tr>
<th>Dependent Variable: ( Q )</th>
<th>Estimate</th>
<th>SE</th>
<th>( t )-Statistic</th>
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<td>( P )</td>
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</tbody>
</table>

Number of Observations 24

Table 10.3: *Beef Demand Regression Results – Linear Model*

\[
\text{Prob[Results IF } H_0 \text{ True]} = \frac{0.0004}{2} \approx 0.0002
\]
Step 5: Decide on the standard of proof, a significance level.

The significance level is the dividing line between the probability being small and the probability being large.

\[
\begin{align*}
\text{Prob[Results IF } H_0 \text{ True]} &< \text{significance level} \quad \text{Prob[Results IF } H_0 \text{ True]} > \text{significance level} \\
\downarrow & \downarrow \\
\text{Prob[Results IF } H_0 \text{ True]} \text{ small} & \text{Prob[Results IF } H_0 \text{ True]} \text{ large} \\
\downarrow & \downarrow \\
\text{Unlikely that } H_0 \text{ is true} & \text{Likely that } H_0 \text{ is true} \\
\downarrow & \downarrow \\
\text{Reject } H_0 & \text{Do not reject } H_0
\end{align*}
\]

We can reject the null hypothesis at the traditional significance levels of 1, 5, and 10 percent. Consequently, the data support the downward sloping demand theory.

A Two-Tailed Test: No Money Illusion Theory

We shall now consider a second theory regarding demand. Microeconomic theory teaches that there is no money illusion; that is, if all prices and income change by the same proportion, the quantity of a good demanded will not change. The basic rationale of this theory is clear. Suppose that all prices double. Every good would be twice as expensive. If income also doubles, however, consumers would have twice as much to spend. When all prices and income doubles, there is no reason for a consumer to change his/her spending patterns; that is, there is no reason for a consumer to change the quantity of any good he/she demands.

We can use indifference curve analysis to motivate this more formally. Recall the household’s utility maximizing problem:

\[
\begin{align*}
\max & \quad \text{Utility} = U(X, Y) \\
\text{s.t.} & \quad P_X X + P_Y Y = I
\end{align*}
\]

A household chooses the bundle of goods that maximizes its utility subject to its budget constraint. How can we illustrate the solution to the household’s problem? First, we draw the budget constraint. To do so, let us calculate its intercepts:
Next, to maximize utility, we find the highest indifference curve that still touches the budget constraint as illustrated in Figure 10.6.

Now, suppose that all prices and income double:

Before

\[
\begin{align*}
\max & \quad \text{Utility} = U(X, Y) \\
\text{s.t.} & \quad P_X X + P_Y Y = I
\end{align*}
\]

After

\[
\begin{align*}
\max & \quad \text{Utility} = U(X, Y) \\
\text{s.t.} & \quad 2P_X X + 2P_Y Y = 2I
\end{align*}
\]
How is the budget constraint affected? To answer this question, calculate the intercepts after all prices and income have doubled and then compare them to the original ones:

\[ 2P_x X + 2P_y Y = 2I \]

\[ \frac{2I}{2P_x} = \frac{I}{P_x} \quad \text{X-intercept: } X = 0 \]

\[ \frac{2I}{2P_y} = \frac{I}{P_y} \quad \text{Y-intercept: } Y = 0 \]

Since the intercepts have not changed, the budget constraint line has not changed; hence, the solution to the household’s constrained utility maximizing problem will not change.

Summary: The no money illusion theory is based on sound logic. But remember, many theories that appear to be sensible turn out to be incorrect. That is why we must test our theories.

Project: Use the beef demand data to assess the no money illusion theory. Can we use our linear demand model to do so? Unfortunately, the answer is no. The linear demand model is inconsistent with the proposition of no money illusion. We shall now explain why.

Linear Demand Model and Money Illusion Theory

The linear demand model is inconsistent with the no money illusion proposition because it implicitly assumes that “slope” of the demand curve equals a constant value, \( \beta_p \), and unaffected by income or the price chicken. To understand why, consider the linear model:

\[ Q = \beta_{Const} + \beta_p P + \beta_I I + \beta_{Chick} P \]

and recall that when we draw a demand curve income and the price of chicken remain constant. Consequently for a demand curve:

\[ Q = Q_{Intercept} + \beta_p P \quad \text{where } Q_{Intercept} = \beta_{Const} + \beta_I I + \beta_{Chick} P \]

This is just an equation for a straight line; \( \beta_p \) equals the “slope” of the demand curve.
Now, consider three different beef prices and the quantity of beef demanded at each of the prices while income and chicken prices remain constant:

<table>
<thead>
<tr>
<th>Price of Beef</th>
<th>$P_0$</th>
<th>$P_1$</th>
<th>$P_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quantity of Beef Demanded</td>
<td>$Q_0$</td>
<td>$Q_1$</td>
<td>$Q_2$</td>
</tr>
</tbody>
</table>

When the price of beef is $P_0$, $Q_0$ units of beef are demanded; when the price of beef is $P_1$, $Q_1$ units of beef are demanded; and when the price of beef is $P_2$, $Q_2$ units of beef are demanded.

Now, suppose that income and the price of chicken doubles. When there is no money illusion:

- $Q_0$ units of beef would still be demanded if the price of beef rises from $P_0$ to $2P_0$.
- $Q_1$ units of beef would still be demanded if the price of beef rises from $P_1$ to $2P_1$.
- $Q_2$ units of beef would still be demanded if the price of beef rises from $P_2$ to $2P_2$. 

Figure 10.7: Demand Curve for Beef
After income and the price of chicken double

<table>
<thead>
<tr>
<th>Price of Beef</th>
<th>Quantity of Beef Demanded</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2P_0$</td>
<td>$Q_0$</td>
</tr>
<tr>
<td>$2P_1$</td>
<td>$Q_1$</td>
</tr>
<tr>
<td>$2P_2$</td>
<td>$Q_2$</td>
</tr>
</tbody>
</table>

Figure 10.8 illustrates this:

Figure 10.8: Demand Curve for Beef and No Money Illusion
We can now sketch in the new demand curve that emerges when income and the price of chicken double. Just connect the points \((Q_0, 2P_0), (Q_1, 2P_1),\) and \((Q_2, 2P_2).\) As Figure 10.9 illustrates, the slope of the demand curve has changed.

![Figure 10.9: Two Demand Curves for Beef – Before and After Income and the Price of Chicken Doubling](image)

But now recall that the linear demand model implicitly assumes that “slope” of the demand curve equals a constant value, \(\beta_P,\) and unaffected by income or the price chicken. Consequently, a linear demand model is intrinsically inconsistent with the existence of no money illusion. We cannot use a model that is inconsistent with the theory to assess the theory. So, we must find a different model.
**Constant Elasticity Demand Model and Money Illusion Theory**

To test the theory of no money illusion, we need a model of demand that can be consistent with it. The constant elasticity demand model is such a model:

\[ Q = \beta_{const} P^{\beta_p} I^{\beta_I} ChickP^{\beta_{CP}} \]

The three exponents equal the elasticities. The beef price exponent equals the own price elasticity of demand, the income exponent equals the income elasticity of demand, and the exponent of the price of chicken equals the cross price elasticity of demand:

\[ \beta_p = \text{(Own) Price Elasticity of Demand} \]
\[ = \text{Percent change in the quantity of beef demanded resulting from} \]
\[ \text{a one percent change in the price of beef (the good’s own price)} \]
\[ = \frac{dQ}{dP} \frac{P}{Q} \]

\[ \beta_I = \text{Income Elasticity of Demand} \]
\[ = \text{Percent change in the quantity of beef demanded resulting from} \]
\[ \text{a one percent change in income} \]
\[ = \frac{dQ}{dI} \frac{I}{Q} \]

\[ \beta_{CP} = \text{Cross Price Elasticity of Demand} \]
\[ = \text{Percent change in the quantity of beef demanded resulting from} \]
\[ \text{a one percent change in the price of chicken} \]
\[ = \frac{dQ}{dChickP} \frac{ChickP}{Q} \]

A little algebra allows us to show that the constant elasticity demand model is consistent with the money illusion theory whenever the exponents sum to 0. Let \( \beta_p + \beta_I + \beta_{CP} = 0 \) and solve for \( \beta_{CP} \):

\[ \beta_p + \beta_I + \beta_{CP} = 0 \]
\[ \downarrow \quad \text{Solving for } \beta_{CP} \]
\[ \beta_{CP} = -\beta_p - \beta_I \]
Now apply this result to the constant elasticity model:

\[ Q = \beta_{\text{Const}} P^{\beta_p} I^{\beta_i} \text{ChickP}^{\beta_{CP}} \]

Substitution for \( \beta_{CP} \)

\[ = \beta_{\text{Const}} P^{\beta_p} I^{\beta_i} \text{ChickP}^{(-\beta_p - \beta_i)} \]

Splitting the exponent

\[ = \beta_{\text{Const}} P^{\beta_p} I^{\beta_i} \text{ChickP}^{-\beta_p} \text{ChickP}^{-\beta_i} \]

Simplifying

\[ = \beta_{\text{Const}} \left( \frac{P^{\beta_p}}{\text{ChickP}^{\beta_p}} \right) \left( \frac{I^{\beta_i}}{\text{ChickP}^{\beta_i}} \right) \]

Moving negative exponents to denominator

\[ = \beta_{\text{Const}} \left( \frac{P}{\text{ChickP}} \right)^{\beta_p} \left( \frac{I}{\text{ChickP}} \right)^{\beta_i} \]

What happens to the two fractions whenever the price of beef (the good’s own price), income, and the price of chicken change by the same proportion? Both the numerators and denominators increase by the same proportion; hence, the fractions remain the same. Therefore, the quantity of beef demanded remains the same. This model of demand is consistent with our theory whenever the exponents sum to 0.

Let us begin the hypothesis testing process. We have already completed Step 0.

**Step 0:** Formulate a model reflecting the theory to be tested.

\[ Q = \beta_{\text{Const}} P^{\beta_p} I^{\beta_i} \text{ChickP}^{\beta_{CP}} \]

**No Money Illusion Theory:** The elasticities sum to 0: \( \beta_p + \beta_i + \beta_{CP} = 0 \).
Step 1: Collect data, run the regression, and interpret the estimates.

Natural logarithms convert the original equation for the constant elasticity demand model into its “linear” form:

\[ \log(Q_t) = \log(\beta_{\text{Const}}) + \beta_p \log(P_t) + \beta_I \log(I_t) + \beta_{CP} \log(\text{Chick}P_t) + e_t \]

To apply the ordinary least squares (OLS) estimation procedure we must first generate the logarithms:

\[ \log(Q_t) = \log(\beta_{\text{Const}}) + \beta_p \log(P_t) + \beta_I \log(I_t) + \beta_{CP} \log(\text{Chick}P_t) + e_t \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ \text{Log}Q_t = \log(\beta_{\text{Const}}) + \beta_p \text{Log}P_t + \beta_I \text{Log}I_t + \beta_{CP} \text{LogChickP}_t + e_t \]

where

\[ \text{Log}Q_t = \log(Q_t) \]
\[ \text{Log}P_t = \log(P_t) \]
\[ \text{Log}I_t = \log(I_t) \]
\[ \text{LogChickP}_t = \log(\text{Chick}P_t) \]

Next, we run a regression with the log of the quantity of beef demanded as the dependent variable; the log of the price of beef (the good’s own price), log of income, and the log of the price of the price of chicken are the explanatory variables:

[Link to MIT-BeefDemand-1985-1986.wf1 goes here.]

### Ordinary Least Squares (OLS)

**Dependent Variable:** LogQ

**Explanatory Variable(s):**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Estimate</th>
<th>SE</th>
<th>t-Statistic</th>
<th>Prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>LogP</td>
<td>-0.411812</td>
<td>0.093532</td>
<td>-4.402905</td>
<td>0.0003</td>
</tr>
<tr>
<td>LogI</td>
<td>0.508061</td>
<td>0.266583</td>
<td>1.905829</td>
<td>0.0711</td>
</tr>
<tr>
<td>LogChickP</td>
<td>0.124724</td>
<td>0.071415</td>
<td>1.746465</td>
<td>0.0961</td>
</tr>
<tr>
<td>Const</td>
<td>9.499258</td>
<td>2.348619</td>
<td>4.044615</td>
<td>0.0006</td>
</tr>
</tbody>
</table>

Number of Observations 24

**Estimated Equation:** \( \text{EstLog}Q = 9.50 - .41 \text{Log}P + .51 \text{Log}I + .12 \text{LogChick} \)

Table 10.4: Beef Demand Regression Results – Constant Elasticity Model

### Interpreting the Estimates

\( b_p = \) Estimate for the (Own) Price Elasticity of Demand = −.41

We estimate that a one percent increase in the price of beef (the good’s own price) decreases the quantity of beef demanded by .41 percent when income and the price of chicken remain constant.

\( b_I = \) Estimate for the Income Elasticity of Demand = .51

We estimate that a one percent increase in income increases the quantity of beef demanded by .51 percent when the price of beef and
the price of chicken remain constant.

\[ b_{CP} = \text{Estimate for the Cross Price Elasticity of Demand} = .12 \]

We estimate that a one percent increase in the price of chicken increases the quantity of beef demanded by .12 percent when the price of beef and income remain constant.

What happens when the price of beef (the good’s own price), income, and the price of chicken increase by one percent simultaneously? The total estimated percent change in the quantity of beef demanded equals sum of the individual changes; that is, the total estimated percent change in the quantity of beef demanded equals the estimated percent change in the quantity demanded resulting from

- a one percent change in the price of beef (the good’s own price) plus
- a one percent change in income plus
- a one percent change in the price of chicken.

The estimated percent change in the quantity demanded equals the sum of the elasticity estimates. We can express this succinctly.

\[
\text{Estimated Percent Change in } Q = b_P + b_I + b_{CP}
\]

\[
\begin{align*}
\text{Estimated Percent Change in } Q &= -0.41 + 0.51 + 0.12 \\
&= 0.22
\end{align*}
\]

A one percent increase in all prices and income results in a .22 percent increase in quantity of beef demanded, suggesting that money illusion is present. As far as the no money illusion theory is concerned, the sign of the elasticity estimate sum is not critical. The fact that the estimated sum is +.22 is not crucial; a sum of −.22 would be just as damning. What is critical is that the sum does not equal 0 as claimed by the money illusion theory.

**Critical Result:** The sum of the elasticity estimates equals .22. The sum does not equal 0; the sum is .22 from 0. This evidence suggests that money illusion is present and the no money illusion theory is incorrect. Since the critical result is that the sum lies .22 from 0, a two-tailed test, rather than a one-tailed test is appropriate.
Step 2: Play the cynic and challenge the results; construct the null and alternative hypotheses.

Cynic’s View: Sure, the elasticity estimates do not sum to 0 suggesting that money illusion exists, but this is just “the luck of the draw.” In fact, money illusion is not present; the sum of the actual elasticities equals 0.

The cynic claims that the .22 elasticity estimate sum results simply from random influences. A more formal way of expressing the cynic’s view is to say that the .22 estimate for the elasticity sum is not statistically different from 0. An estimate is not statistically different from 0 whenever the nonzero results from random influences.

Let us now construct the null and alternative hypotheses:

\[ H_0: \beta_p + \beta_I + \beta_{CP} = 0 \] Cynic’s is correct: Money illusion not present

\[ H_1: \beta_p + \beta_I + \beta_{CP} \neq 0 \] Cynic’s is incorrect: Money illusion present

The null hypothesis, like the cynic, challenges the evidence. The alternative hypothesis is consistent with the evidence. Can we dismiss the cynic’s view as nonsense?

Econometrics Lab 10.1: Could the Cynic Be Correct?

We shall use a simulation to show that the cynic could indeed be correct. In this simulation, Coef1, Coef2, and Coef3 denote the coefficients for the three explanatory variables. By default, the actual values of the coefficients are \(-.5\), .4, and .1. The actual values sum to 0.
Be certain that the Pause checkbox is checked. Click Start. The coefficient estimates for each of the three coefficients and their sum are reported:

- The coefficient estimates do not equal their actual values.
- The sum of the coefficient estimates does not equal 0 even though the sum of the actual coefficient values equals 0.

Click Continue a few more times. As a consequence of random influences, we could never expect the estimate for an individual coefficient to equal its actual value. Therefore, we could never expect a sum of coefficient estimates to equal the sum of their actual values. Even if the actual elasticities sum to 0, we could never expect the sum of their estimates to equal precisely 0. Consequently, we cannot dismiss the cynic’s view as nonsense.

**Step 3:** Formulate the question to assess the cynic’s view and the null hypothesis.

- **Generic Question:** What is the probability that the results would be like those we actually obtained (or even stronger), if the cynic is correct and money illusion was not present?
- **Specific Question:** The sum of the coefficient estimates is .22 from 0. What is the probability that the sum of the coefficient estimates in one regression would be .22 or more from 0, if H₀ were true (if the sum of the actual elasticities equaled 0)?

**Answer:** Prob[Results IF H₀ True]

<table>
<thead>
<tr>
<th>Prob[Results IF H₀ True] small</th>
<th>Prob[Results IF H₀ True] large</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unlikely that H₀ is true</td>
<td>Likely that H₀ is true</td>
</tr>
<tr>
<td>Reject H₀</td>
<td>Do not reject H₀</td>
</tr>
<tr>
<td>Estimate sum is statistically different from 0</td>
<td>Estimate sum is not statistically different from 0</td>
</tr>
</tbody>
</table>

**Step 4:** Use the general properties of the estimation procedure, the probability distribution of the estimate, to calculate Prob[Results IF H₀ True].

How can we calculate this probability? We shall explore three approaches that can be used:

- Clever algebraic manipulation
- Wald (F-distribution) test
- Letting statistical software do the work
Calculating Prob[Results IF $H_0$ True]: Clever Algebraic Manipulation

We begin with the clever algebraic manipulation approach. This approach exploits the tails probability reported in the regression printout. Recall that the tails probability is based on the premise that the actual value of the coefficient equals 0. Our strategy takes advantage of this:

- First, cleverly define a new coefficient that equals 0 when the null hypothesis is true.
- Second, reformulate the model to incorporate the new coefficient.
- Third, use the ordinary least squares (OLS) estimation procedure to estimate the parameters of the new model.
- Then, focus on the estimate of the new coefficient. Use the new coefficient estimate’s tails probability to calculate Prob[Results IF $H_0$ True].

**Step 0:** Formulate a model reflecting the theory to be tested.

Begin with the null and alternative hypotheses:

- $H_0$: $\beta_p + \beta_t + \beta_{CP} = 0$  Cynic is correct: Money illusion not present
- $H_1$: $\beta_p + \beta_t + \beta_{CP} \neq 0$  Cynic is incorrect: Money illusion present

Now, cleverly define a new coefficient so that the null hypothesis is true when the new coefficient equals 0:

$$\beta_{Clever} = \beta_p + \beta_t + \beta_{CP}$$

Clearly, $\beta_{Clever}$ equals 0 if and only if the elasticities sum to 0 and no money illusion exists; that is, $\beta_{Clever}$ equals 0 if and only if the null hypothesis is true.
Now, we shall use algebra to reformulate the constant elasticity of demand model to incorporate $\beta_{\text{Clever}}$:

$$\log(Q_t) = \log(\beta_{\text{Const}}) + \beta_p \log(P_t) + \beta_I \log(I_t) + \beta_{CP} \log(\text{ChickP}_t) + e_t$$

Solving for $\beta_{CP}$:

$$\beta_{CP} = \beta_{\text{Clever}} - \beta_P - \beta_I.$$ 

Substitute for $\beta_{CP}$:

$$\log(Q_t) = \log(\beta_{\text{Const}}) + \beta_p \log(P_t) + \beta_I \log(I_t) + \beta_{Clever} \log(\text{ChickP}_t) - \beta_P \log(\text{ChickP}_t) - \beta_I \log(\text{ChickP}_t) + e_t$$

Multiplying $\log(\text{ChickP}_t)$ term.

$$\log(Q_t) = \log(\beta_{\text{Const}}) + \beta_p \log(P_t) - \beta_P \log(\text{ChickP}_t) + \beta_I \log(I_t) - \beta_I \log(\text{ChickP}_t) + \beta_{Clever} \log(\text{ChickP}_t) + e_t$$

Rearranging terms.

$$\log(Q_t) = \log(\beta_{\text{Const}}) + \beta_p [\log(P_t) - \log(\text{ChickP}_t)] + \beta_I [\log(I_t) - \log(\text{ChickP}_t)] + \beta_{Clever} \log(\text{ChickP}_t) + e_t$$

Defining new variables.

$$\log(Q_t) = \log(\beta_{\text{Const}}) + \beta_P \log(P_t) - \beta_P \log(\text{ChickP}_t) + \beta_I \log(I_t) - \beta_I \log(\text{ChickP}_t) + \beta_{Clever} \log(\text{ChickP}_t) + e_t$$

where

- $\log(Q_t) = \log(Q_t)$
- $\log(P_t) - \log(\text{ChickP}_t) = \log(P_t) - \log(\text{ChickP}_t)$
- $\log(I_t) - \log(\text{ChickP}_t) = \log(I_t) - \log(\text{ChickP}_t)$
- $\log(\text{ChickP}_t) = \log(\text{ChickP}_t)$
Step 1: Collect data, run the regression, and interpret the estimates.

Now, use the ordinary least squares (OLS) estimation procedure to estimate the parameters of this model:

**Ordinary Least Squares (OLS)**

**Dependent Variable:** $\log Q$

<table>
<thead>
<tr>
<th>Explanatory Variable(s)</th>
<th>Estimate</th>
<th>SE</th>
<th>t-Statistic</th>
<th>Prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log P \text{LessLogChick}$</td>
<td>-0.411812</td>
<td>0.093532</td>
<td>-4.402905</td>
<td>0.0003</td>
</tr>
<tr>
<td>$\log I \text{LessLogChick}$</td>
<td>0.508061</td>
<td>0.266583</td>
<td>1.905829</td>
<td>0.0711</td>
</tr>
<tr>
<td>$\log Chick$</td>
<td>0.220974</td>
<td>0.275863</td>
<td>0.801027</td>
<td>0.4325</td>
</tr>
<tr>
<td><strong>Const</strong></td>
<td>9.499258</td>
<td>2.348619</td>
<td>4.044615</td>
<td>0.0006</td>
</tr>
</tbody>
</table>

Number of Observations 24

**Estimated Equation:** $\hat{\log Q} = 9.50 - 0.41 \log P \text{LessLogChick}$  
  $+ 0.51 \log I \text{LessLogChick} + 0.22 \log Chick$

**Critical Result:** $b_{\text{Clever}}$, the estimate for the sum of the actual elasticities, equals .22. The estimate does not equal 0; the estimate is .22 from 0.

This evidence suggests that money illusion is present and the no money illusion theory is incorrect.

Table 10.5: Beef Demand Regression Results – Constant Elasticity Model

It is important to note that the estimates of the reformulated model are consistent with the estimates of the original model:

- The estimate of the price coefficient is the same in both cases, $-0.41$.
- The estimate of the income coefficient is the same in both cases, $0.51$.
- In the reformulated model, the estimate of $\beta_{\text{Clever}}$ equals $0.22$ which equals the sum of the elasticity estimates in the original model.

Step 2: Play the cynic and challenge the results; reconstruct the null and alternative hypotheses.

**Cynic’s View:** Sure, $b_{\text{Clever}}$, the estimate for the sum of the actual elasticities, does not equal 0 suggesting that money illusion exists, but this is just “the luck of the draw.” In fact, money illusion is not present; the sum of the actual elasticities equals 0.

We now reformulate the null and alternative hypotheses in terms of $\beta_{\text{Clever}}$:

- $H_0: \beta_p + \beta_I + \beta_{CP} = 0 \quad \Rightarrow \quad \beta_{\text{Clever}} = 0 \quad \text{Cynic is correct: Money illusion not present}$
- $H_1: \beta_p + \beta_I + \beta_{CP} \neq 0 \quad \Rightarrow \quad \beta_{\text{Clever}} \neq 0 \quad \text{Cynic is incorrect: Money illusion present}$
We have already showed that we cannot dismiss the cynic’s view as nonsense. As a consequence of random influences, we could never expect the estimate for $\beta_{\text{Clever}}$ to equal precisely 0, even if the actual elasticities sum to 0.

**Step 3:** Formulate the question to assess the cynic’s view and the null hypothesis.

![Figure 10.11: Probability Distribution of the Clever Coefficient Estimate](image)

- **Generic Question:** What is the probability that the results would be like those we obtained (or even stronger), if the cynic is correct and no money illusion was present?
- **Specific Question:** What is the probability that the coefficient estimate in one regression, $b_{\text{Clever}}$, would be at least .22 from 0, if $H_0$ were true (if the actual coefficient, $\beta_{\text{Clever}}$, equals 0)?

**Answer:**

<table>
<thead>
<tr>
<th>Prob[Results IF $H_0$ True] small</th>
<th>Prob[Results IF $H_0$ True] large</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unlikely that $H_0$ is true</td>
<td>Likely that $H_0$ is true</td>
</tr>
<tr>
<td>Reject $H_0$</td>
<td>Do not reject $H_0$</td>
</tr>
<tr>
<td>$b_{\text{Clever}}$ and the estimate sum is statistically different from 0</td>
<td>$b_{\text{Clever}}$ and the estimate sum is not statistically different from 0</td>
</tr>
</tbody>
</table>
Step 4: Use the general properties of the estimation procedure, the probability distribution of the estimate, to calculate Prob[Results IF $H_0$ True].

![Student t-distribution](image)

**Figure 10.12: Calculating Prob[Results IF $H_0$ True]**

<table>
<thead>
<tr>
<th>OLS estimation</th>
<th>If $H_0$</th>
<th>Standard error</th>
<th>Number of observations</th>
<th>Number of parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>procedure unbiased</td>
<td>true</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Mean[$b_{Clever}$] = $\beta_{Clever}$ = 0

SE[$b_{Clever}$] = .2759

DF = 24 - 4 = 20

The software automatically computes the tails probability based on the premise that the actual value of the coefficient equals 0. This is precisely what we need, is it not? The regression printout reports that the tails probability equals .4325. Consequently,

Prob[Results IF $H_0$ True] = .4325.
Step 5: Decide on the standard of proof, a significance level.

The significance level is the dividing line between the probability being small and the probability being large.

<table>
<thead>
<tr>
<th>Prob[Results IF H₀ True] Less Than Significance Level</th>
<th>Prob[Results IF H₀ True] Greater Than Significance Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prob[Results IF H₀ True] small</td>
<td>Prob[Results IF H₀ True] large</td>
</tr>
<tr>
<td>Unlikely that H₀ is true</td>
<td>Likely that H₀ is true</td>
</tr>
<tr>
<td>Reject H₀</td>
<td>Do not reject H₀</td>
</tr>
</tbody>
</table>

bₐClever and the estimate sum is statistically different from 0

The probability exceeds the traditional significance levels of 1, 5, and 10 percent. Based on the traditional significance levels, we would not reject the null hypothesis. We would conclude that bₐClever and the estimate sum is not statistically different from 0, thereby supporting the no money illusion theory.

In the next chapter we shall explore two other ways to calculate Prob[Results IF H₀ True].

---

1 If you are not familiar with indifference curve analysis, please skip to the Linear Demand Model and Money Illusion Theory section and accept the fact that the no money illusion theory is well grounded in economic theory.

2 Again recall that quantity is plotted on the horizontal and price is plotted on the vertical axis, the slope of the demand curve is actually the reciprocal of βₚ, 1/βₚ. That is why we place the word slope within quotes. This does not affect the validity of our argument, however. The important point is that the linear model implicitly assumes that the slope of the demand curve is constant, unaffected by changes in other factors relevant to demand.