6.17 ** Find the geodesics on the cone whose equation in cylindrical polar coordinates is \( z = \lambda \rho \). [Let the required curve have the form \( \phi = \phi(\rho) \).] Check your result for the case that \( \lambda \to 0 \).

Geodesics on a cone? \((\text{let curve have form } \phi = \phi(\rho))\)

\[ z = \lambda \rho \]

In cylindrical coords,
- a small move in \( z \)-direction adds dist. \( \Delta z \)
- a small move in \( \rho \)-dir. adds dist. \( \Delta \rho \)
- a small move in \( \phi \)-dir. adds dist. \( \rho \Delta \phi \)

So:

\[ \Delta s = \sqrt{(\Delta z)^2 + (\Delta \rho)^2 + (\rho \Delta \phi)^2} \]

In cylindrical coords.

Using \( z = \lambda \rho \) \( \Rightarrow \Delta z = \lambda \Delta \rho \), we have

\[ \Delta s = \sqrt{\lambda^2 (\Delta \rho)^2 + (\Delta \rho)^2 + (\rho \Delta \phi)^2} \]

\[ = \Delta \rho \sqrt{(\lambda^2 + 1) + \lambda (\frac{\Delta \phi}{\rho})^2} \]

\[ = \Delta \rho \sqrt{(\lambda^2 + 1) + \rho \frac{\Delta \phi}{\rho}} \]

- A geodesic should extremize distance, i.e.

\[ L = \int_{\rho_i}^{\rho_f} \Delta s = \int_{\rho_i}^{\rho_f} \rho \sqrt{(\lambda^2 + 1) + \rho \frac{\Delta \phi}{\rho}} = 0 \]

should be stationary for geodesic.

We can apply the Euler-Lagrange equations:

\[ \frac{\partial f}{\partial \rho} - \frac{d}{d \rho} \frac{\partial f}{\partial \rho'} = 0 \]

where \( f[\rho, \rho'; \phi] = \sqrt{(\lambda^2 + 1) + \rho \frac{\Delta \phi}{\rho}} \)

observe:

\[ \frac{\partial f}{\partial \phi} = 0 \]

\[ \frac{d}{d \rho} \frac{\partial f}{\partial \rho'} = 0 \Rightarrow \frac{\partial f}{\partial \rho'} = C \]
\[ c = \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial \phi'} \left[ (\lambda^2 + 1) + \rho^2 (\phi')^2 \right]^{1/2} \]

\[ = \frac{1}{2} \frac{\rho}{\left[ (\lambda^2 + 1) + \rho^2 (\phi')^2 \right]^{1/2}} \rho^2 \phi' \]

\[ c = \frac{\rho^2 \phi'}{\left[ (\lambda^2 + 1) + \rho^2 (\phi')^2 \right]^{1/2}} \]

\[ c^2 \left[ (\lambda^2 + 1) + \rho^2 (\phi')^2 \right] = \rho \phi'^2 \]

\[ (\phi')^2 \left[ \rho^2 - c^2 \phi^2 \right] = c^2 (\lambda^2 + 1) \]

\[ (\phi')^2 \rho^2 \left[ \rho^2 - c^2 \right] = c^2 (\lambda^2 + 1) \]

\[ (\phi')^2 = \frac{c^2 (\lambda^2 + 1)}{\rho^2 \left[ \rho^2 - c^2 \right]} \]

\[ \phi' = \frac{c}{\rho \sqrt{\rho^2 - c^2}} \]

\[ \phi(p) = \phi_0 + \frac{c}{\rho \sqrt{\rho^2 - c^2}} \int dp \frac{c}{\rho \sqrt{\rho^2 - c^2}} \]

\[ \text{Interpret \ constant} \quad \rho = c x \quad \frac{dp}{dx} = c \ dx \]

\[ = \phi_0 + \frac{c}{\sqrt{\lambda^2 + 1}} \int dx \frac{1}{x \sqrt{\lambda^2 + 1}} \]

\[ \phi(p) = \phi_0 + \frac{c}{\sqrt{\lambda^2 + 1}} \cos^{-1} \left( \frac{\lambda^2}{\rho} \right) \]

\[ \phi(p) = \phi_0 + \frac{c}{\sqrt{\lambda^2 + 1}} \cos^{-1} \left( \frac{\rho}{c} \right) \]

For \( \lambda \to 0 \), i.e. the line approaches the \( x-y \) plane \((z \to 0)\)

\[ \phi(p) = \phi_0 + \cos^{-1} \left( \frac{c}{p} \right) \]

\[ \cos (\phi(p) - \phi_0) = \frac{c}{p} \]

\[ p = \frac{c}{\cos (\phi(p) - \phi_0)} \]

\[ p \cos (\phi(p) - \phi_0) = c \]

What does this look like?

If \( \phi_0 = 0 \), i.e. \( p \cos \phi = c \), we'd recognize it as \( \phi = \arctan \frac{y}{x} \)
\[ \rho \cos \phi = c, \]

a straight line perpendicular to \( \phi = 0 \), passing a minimum distance \( c \) from the origin.

For \( \phi_0 \neq 0 \), this is a straight line \( \perp \) to the \( \phi = \phi_0 \) line, passing a minimum distance \( c \) from the origin.

\[ \rho \cos(\beta - \phi_0) = c \]