As the Planimeter's Wheel Turns

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A classic example of Green's Theorem in action is the planimeter, a device that measures the area enclosed by a curve. Most familiar may be the polar planimeter (see Figure 1), for which a nice geometrical explanation is given in [1] and a direct constructive proof using Green's Theorem in [2]. Another type is the "rolling planimeter", which is particularly suitable in a vector calculus course for both ease of use and simplicity of proof. In this article, we present very simple proofs using Green's Theorem for both types of planimeter. These proofs are suitable for use in a vector calculus course and avoid the awkward trigonometric and algebraic calculations required by proofs like that in [2] and reveal more clearly the role of Green's Theorem. Both rolling and polar planimeters are available in modern mechanical and electronic versions for commercial use (a quick web search reveals several manufacturers). For classroom demonstrations, relatively inexpensive planimeters are available on eBay; unfortunately, rolling planimeters can be harder to find than polar planimeters.

Before embarking on an analysis of how planimeters work, let us take a moment to reflect on how marvelous these instruments really are in their simplicity of design and effectiveness.

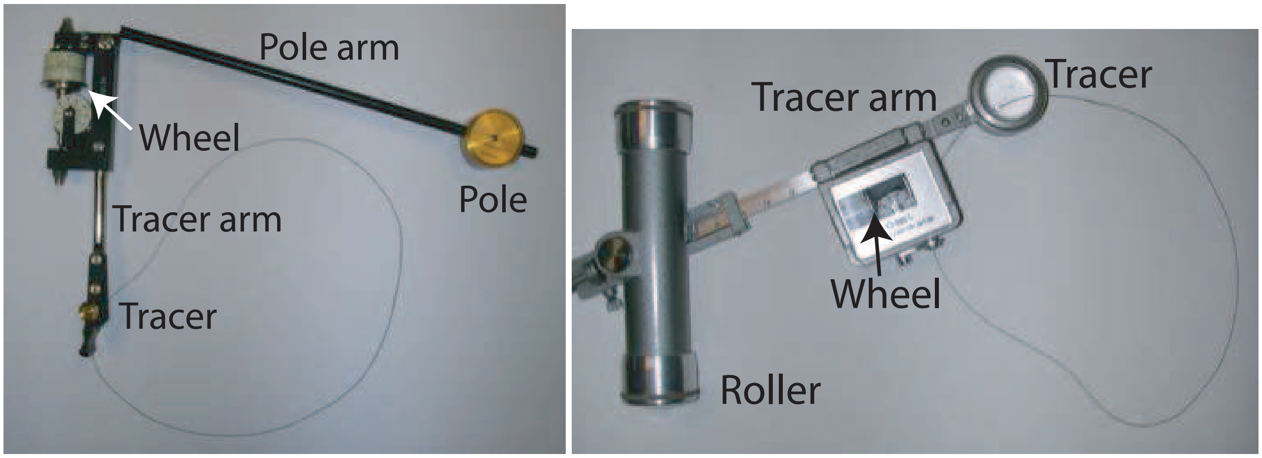

Figure 1: A Keuffel and Esser polar planimeter on the left and an Ushikata rolling planimeter on the right (part of author's collection).

1 Green's Theorem of the mind?

Consider the task of judging the area of the state of Maine on a map. After studying Riemann sums, one might take a ruler, divide the state up into small squares, and then add up the areas of the squares. The error would depend on "curviness" of the boundary, the size of the squares, and the accuracy of the ruler. For example, the area of a rectangle can be measured quite precisely in this manner (up to the accuracy of the ruler), while measuring an area bounded by a coast would be more difficult to calculate.

The opposite of this "brute-force" approach would be to choose a tool more appropriate for the task: a planimeter. The planimeter's accuracy does not depend on the complexity of the boundary, and it will *directly* measure the desired area with no need for any computations or analysis. One simply runs the tracer of the planimeter around the curve and then reads off the value on the vernier that records how much the wheel rolled. The accuracy depends on the steadiness of one's hand and the quality of the instrument. No area calculations are involved! On first seeing a planimeter, one might be skeptical that it is truly finding the area, since it makes no direct area measurements.

This surprising feature of the planimeter has an analog in human perception. When we do tasks like estimating area on a map or catching a ball, what are we really perceiving? It seems unlikely that we do a brute-force approach like imagining a grid superimposed on the paper. Suppose you are driving a car on a busy road. If the car in front of you slows, you need to judge how much time you have to stop to avert a fender-bender. As you drive, do you judge the relative distance, speed, and acceleration of the other cars? If you have to determine all of these quantities separately and then calculate from them the stopping time (like a classic calculus problem!), driving would be a constant mental effort. Some psychologists [3] argue that we are more like planimeters: we are very good at certain tasks because we are designed to perceive the most effective information, just as the planimeter does not calculate area in the most obvious way but rather in a more subtle and most effective manner, made possible by Green's Theorem. What theorem paves the way for hitting baseballs with bats and driving fast cars on congested freeways? Perception/action researchers continue to seek answers to this question.

Now let us return to our task of analyzing the planimeter. First we treat the simpler case of a rolling planimeter, then in the following section we examine the action of a polar planimeter.

2 Rolling planimeter

Let C be a positively oriented, piecewise smooth, simple closed curve. Recall that Green's Theorem states that, given functions $P(x, y)$ and $Q(x, y)$ whose partial derivatives are continuous on an open set containing the region R enclosed by the curve C, we have

$$\int_C P\ dx + Q\ dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA. \tag{1}$$

In particular, apply Green's Theorem with $P = 0, Q = x$, and then apply again with $P = -y, Q = 0$, to obtain the relations

$$\int_C x\ dy = -\int_C y\ dx = \iint_R dA = \text{Area of the region } R. \tag{2}$$

We will use the coordinates (x, y) for points on the curve C, while $(0, Y)$ will describe the position of the pivot of the rolling planimeter. See Figure 2. It is crucial to recognize that as the pointer traces out the curve C, the planimeter's roller can only roll forward and backward (the roller cannot turn), so it is as though the pivot were fixed to a straight line which we will make our y-axis. As the planimeter traces out the curve, the roller moves up and down the y-axis, while the tracer arm rotates on the pivot. Hence the rolling planimeter is really a form of linear planimeter (as opposed to a polar planimeter whose pivot traces out a circular arc–see the next section). Also note that the tracer arm's rotation is limited and it may not swing past the roller.

Consider the motion of the tracer as it moves along a small portion of the curve C, from a point (x, y) counterclockwise to $(x + dx, y + dy)$. The pivot will have a corresponding displacement from position $(0, Y)$ to a new position $(0, Y + dY)$. We wish to determine how much the measuring wheel on the tracer arm will turn as a result of this small motion, which can be decomposed into two parts. First roll the pivot along the y-axis from position $(0, Y)$ to $(0, Y + dY)$ so that the tracer arm maintains a fixed angle α with the y-axis and the tracer ends up at $(x, y + dY)$. Next rotate the tracer arm by an angle $d\theta$ (without moving the roller) so that the tracer ends up at $(x + dx, y + dy)$. During this operation, the wheel on the tracer arm will cover a distance of $\sin\alpha\ dY + a\ d\theta = \frac{x}{L}\ dY + a\ d\theta$, since only the component of the motion perpendicular to the tracer arm will result in the wheel turning. The planimeter returns to its original placement after

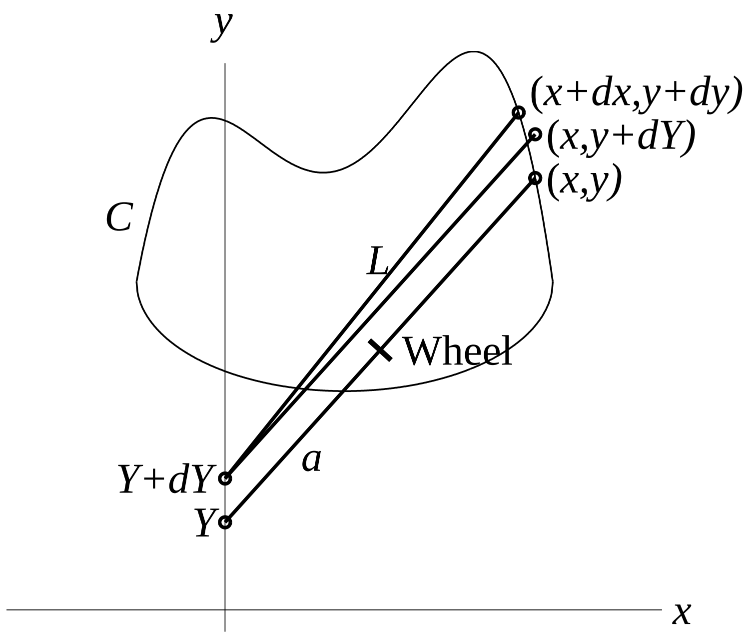

Figure 2: The motion of the tracer arm of a rolling planimeter as it traverses a curve C counter-clockwise. The tracer arm is attached to a roller which rolls along the y-axis. The tracer arm may pivot where it attaches to the roller at $(0, Y)$ as the tracer at its opposite end traces out the curve with coordinates (x, y) .

traversing C and so the total angle of rotation of the tracer arm will be zero. Therefore the total rolling distance of the tracer arm wheel is

$$\text{Total wheel roll} = \frac{1}{L} \oint_{C'} x \; dY, \tag{3}$$

where C' is the curve described by (x, Y), which will be a piecewise smooth, simple closed curve due to the limited rotational angle of the pivot on the roller. To see this, suppose C' intersects itself at some point (x, Y). There are only two possible values of y that can correspond to fixed values of x and Y: $y = Y \pm \sqrt{L^2 - x^2}$. But $(x, Y + \sqrt{L^2 - x^2})$ and $(x, Y - \sqrt{L^2 - x^2})$ cannot both lie on the curve C since this would require the tracer arm to rotate past the roller, which it is unable to do. Hence C' cannot intersect itself.

The distance recorded by the wheel can also be calculated using the scalar projection of the change

(dx, dy) onto a unit vector perpendicular to the tracer arm: $(dx, dy) \cdot (-y + Y, x)/L$. Integrating this around the curve leads to an alternate expression:

$$\text{Total wheel roll} = \frac{1}{L}\oint_C x\ dy - y\ dx + \frac{1}{L}\oint_{C'} Y\ dx = \frac{2}{L}\oint_C x\ dy - \frac{1}{L}\oint_{C'} x\ dY, \tag{4}$$

where we have made use of (2) to convert $-\oint_C y\ dx$ to $\oint_C x\ dy$ and $\oint_{C'} Y\ dx$ to $-\oint_{C'} x\ dY$.

Setting the expression in (3) equal to that in (4), we find that

$$\oint_C x\ dy = \oint_{C'} x\ dY. \tag{5}$$

Referring again to (2) leads to the conclusion that

$$\text{Area of the region } R = L \times (\text{Total wheel roll}). \tag{6}$$

As a practical matter, many planimeters have adjustable length arms as a way to account for the scale of graph, which could be part of a map or pressure chart. According to our formula, doubling the length L cuts the vernier reading of the wheel roll in half, while changing the position of the wheel on the tracer arm (i.e., the length a) does not affect the reading at all. For example, a tracer arm length of 15 cm on the rolling planimeter shown in Figure 1 leads to a vernier reading of 1 corresponding to 100 cm^2. Extending the arm to 30 cm leads to a vernier reading of 1 corresponding to 200 cm^2.

3 Polar planimeter

A proof for the polar planimeter may be constructed by replacing the coordinates $(0, Y)$ with the coordinates $(b\cos\phi, b\sin\phi)$ to reflect the circular motion of the pivot. Consider the motion of the tracer of the polar planimeter as it moves along a small portion of the curve C, from a point (x, y) counterclockwise to $(x + dx, y + dy)$. The pivot will have a corresponding displacement from position $(b\cos\phi, b\sin\phi)$ to a new position $(b\cos(\phi + d\phi), b\sin(\phi + d\phi))$. Since we are considering an infinitesimal displacement, we may linearize the new coordinates to be $(b\cos(\phi) - b\sin(\phi)d\phi), b\sin(\phi) + \cos(\phi)d\phi))$. As before we decompose this small motion into two parts. First swing the pivot along the arc to its new position, keeping the tracer arm parallel to its original orientation and moving the tracer to $(x - b\sin(\phi)d\phi, y + b\cos(\phi)d\phi)$. See Figure 3. Next rotate the tracer arm by an angle $d\theta$ (without changing the pivot's position) so that the tracer ends up at $(x + dx, y + dy)$. During this operation, the wheel on the tracer arm will cover a distance of $\frac{1}{L}(b\sin\phi - y, x - b\cos\phi)\cdot(-b\sin\phi, b\cos\phi)d\phi + a\ d\theta = \frac{b}{L}(x\cos\phi + y\sin\phi - b)d\phi + a\ d\theta$, since only the component of the motion perpendicular to the tracer arm will result in the wheel turning. The planimeter returns to its original placement after traversing C (and cannot do a complete rotation of 360 degrees), so the total angle θ of rotation of the tracer arm will be zero, as will be the total angle ϕ of rotation of the pole arm. Therefore the total rolling distance of the tracer arm wheel is

$$\text{Total wheel roll} = \frac{b}{L}\oint_{C_1'} x\cos\phi\ d\phi + \frac{b}{L}\oint_{C_2'} y\sin\phi\ d\phi, \tag{7}$$

where C_1' is the curve described by (x, ϕ) and C_2' is the curve described by (ϕ, y). Because the pivot has less than 180 degrees of motion (it cannot bend backwards), both C_1' and C_2' are simple closed curves.

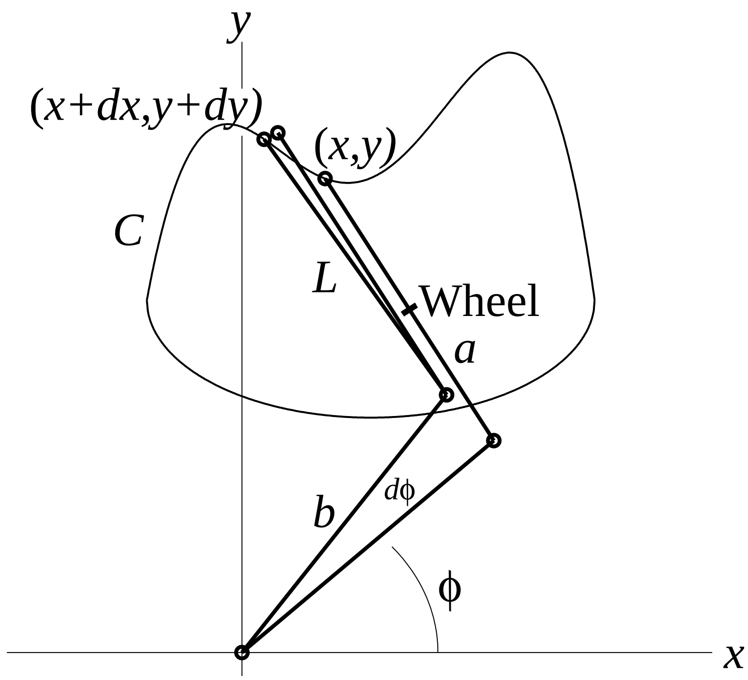

Figure 3: The motion of the tracer arm of a polar planimeter as it traverses a curve C counter-clockwise. The tracer arm is attached via a pivot to an arm with a fixed pole. This pivot traces out a circular arc of radius b as the planimeter traces out the curve.

The distance recorded by the wheel can also be calculated using the scalar projection of the change (dx, dy) onto a unit vector perpendicular to the tracer arm: $(dx, dy) \cdot (b\sin\phi - y, x - b\cos\phi)/L$. Integrating this around the curve leads to an alternate expression:

$$\text{Total wheel roll} = \frac{1}{L}\oint_C x\ dy - y\ dx + \frac{b}{L}\oint_{C_1'} \sin\phi\ dx - \frac{b}{L}\oint_{C_2'} \cos\phi\ dy \tag{8}$$

$$= \frac{2}{L}\oint_C x\ dy - \frac{b}{L}\oint_{C_1'} x\cos\phi\ d\phi - \frac{b}{L}\oint_{C_2'} y\sin\phi\ d\phi. \tag{9}$$

Setting the expression in (7) equal to that in (9), we find that

$$\oint_C x\ dy = b\oint_{C_1'} x\cos\phi\ d\phi + b\oint_{C_2'} y\sin\phi\ d\phi. \tag{10}$$

Referring again to (2) leads to the conclusion that for a polar planimeter, as for a rolling planimeter,

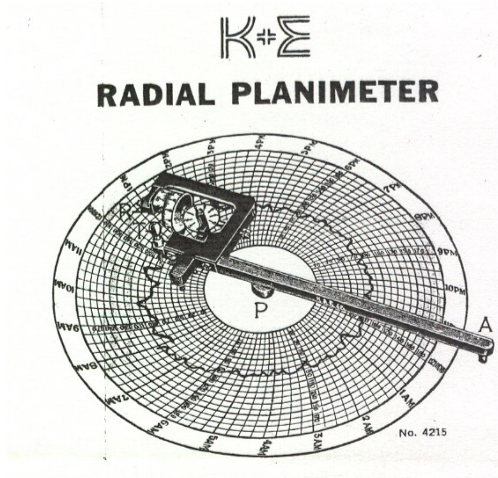

Figure 4: Part of the instructions to a Keuffel and Esser radial planimeter in the author's collection. P marks the pin and T marks the tracer. The tracer arm AT can slide as well as turn on the pin.

we have the wonderful relation

$$\text{Area of the region } R = L \times (\text{Total wheel roll}). \tag{11}$$

4 Radial planimeter

For a bit of a challenge, construct a proof of how a radial planimeter works. A radial planimeter measures the mean height of circular diagrams and consists simply of a tracer arm with a pin, a tracer and a measuring wheel. As the tracer follows the curve, the tracer arm pivots on the pin, which runs along a track inside the tracer arm, allowing the tracer arm to slide back and forth as needed to trace out the curve. See Figure 4. The planimeter rotates completely around the pin after doing a full circuit of the circular diagram.

To explore deeper into planimeters, see the excellent website [4], which happens to include a proof for the radial planimeter.

References

[1] G. Jennings, *Modern Geometry with Applications*, Springer-Verlag, New York, 1985.

[2] R.W. Gatterdam, The planimeter as an example of Green's Theorem, *American Mathematical Monthly 88:9* (1981), 701–704.

[3] S. Runeson, On the possibility of "smart" perceptual mechanisms, *Scand. J. Psychol. 18*, (1977), 172-179.

[4] R. Foote, http://persweb.wabash.edu/facstaff/footer/planimeter/PLANIMETER.HTM.