Math 365 Random Walk on a Circle

The objective of this lab is to continue exploring behavior of Markov chains, building on an example from class. Feel free to discuss problems with each other during lab, as well as ask me questions. Please turn in your solutions by class on Monday 2/22. You don't need to email your R script unless you do non-routine calculations. (*Problems adapted from Ross's Introduction to Probability Models.*)

1 Warm-up problem

Every time the Titans team wins a game, it wins its next game with probability 0.8; every time it loses a game, it wins its next game with probability 0.3 (only win or lose, no ties). If the team wins a game, then it has dinner together with probability 0.7, while if the team loses then it has dinner together with probability 0.2. What proportion of games result in a team dinner?

2 Further analysis of random walk on a circle

A particle moves among N vertices $\{0, 1, 2, \cdots, N-1\}$ arranged on a circle (increasing numbers go counterclockwise). At each step, it moves either one step counterclockwise with probability p or one step clockwise with probability $q = 1 - p$. Assume $X_0 = 0$.

1. Run a simulation:

```
n <- 100
nvertices <- 20
p <- 0.5
coinflips <- sample(c(-1,1),n,replace=TRUE,c(1-p,p))
walk <- cumsum(c(0,coinflips)) %% nvertices # modulo number of edges
theta <- seq(0,2*pi,.01)
for (k in 1:n) {
  plot(cos(theta),sin(theta),type="l",ylim=c(-1.2,1.2),
    asp=1,col="blue",xlab="",ylab="",main=paste("Time step",k))
    points(cos(walk[k]*2*pi/nvertices),sin(walk[k]*2*pi/nvertices),col="red",pch=19)
  Sys.sleep(0.1) }
```

2. Experiment with running the simulation for different values of p. Is the particle more or less likely to make it around the circle as p increases from 1/2 toward 1?

3. Verify that $\bar{\pi}(i) = 1/N$ for all vertices i (check that $\mathbf{vP} = \mathbf{v}$ for $\mathbf{v} = [\frac{1}{N} \cdots \frac{1}{N}]$).

4. Let T be the time of first return to state 1. Find the expected value $\mathbb{E}(T)$.

5. Find the probability that all other positions are visited before the particle returns to its starting state at time T, for cases $p = \frac{1}{2}$ and $p \neq \frac{1}{2}$. (Break down into sum of conditional probabilities based on 1st step, and use gambler's ruin result.)

6. For values $p = 0.5$, 0.6, 0.9, with $N = 20$, calculate the probability you found in the previous exercise. What happens as p increases toward 1? Why does this make sense?