Math 365 Exam 3 (Take-home)

Please answer each problem as clearly and completely as you can. Do not discuss these problems with other students, or anyone else but me. You may use your textbook, lecture notes, class materials (including those posted on the website), and homework, but do not use other books, the internet, or any materials other than those directly associated with the course. Please do feel free to ask me questions, either via email or coming by my office. Show all work to demonstrate that you understand your answer. You may use programs like R, Mathematica, and Wolfram Alpha for computations.

Exam is due 4pm Thursday May 12. Late submissions will be penalized 10 points per day. Problems 1-5 may be submitted either on paper or via email. Please email me your R script for Problem 6.

Problem 1 (20pt) Evaluate the following integrals using Itô's formula (see top of page 209 and bottom of page 212 for the relevant formulas):

$$\int_0^t s\,dW_s \qquad \text{and} \qquad \int_0^t W_s^2\,dW_s$$

Note that both answers will involve $\int_0^t W_s\,ds$, which can't be further simplified (it's easy to compute once you have a particular instance of W_t, but you can't calculate it *a priori*).

Problem 2 (15pt) Let W_t be a standard Brownian motion, and T_a be the time at which W_t first hits a. Calculate the probability that $T_1 < T_{-1} < T_2$, that is, that the path hits 1 before hitting -1, and hits -1 before hitting 2.

Problem 3 (15pt) Suppose the price S_t of a stock can be modeled as a standard Brownian motion W_t (where t is measured, say, in hours and S_t in dollars) plus an initial price b: $S_t = W_t + b$ for $t \geq 0$. The idea is that we expect the price here to fluctuate randomly about b. You decide to sell the stock when its price reaches $b+1$ or in five hours, whichever happens first. What is the probability the price hits $b+1$ within 5 hours?

Problem 4 (15pt) Let W_t be a standard Brownian motion and define $Y_t = W_t^2 - t$. Show that $E[Y_t \,|\, W_s] = Y_s$ for all $t > s \geq 0$. (In fact, Y_t is a martingale, but you don't need to prove the other properties.)

Problem 5 (15pt) Let N_t be a Poisson process for $t \geq 0$ with rate parameter λ, and define $M_t = N_t - \lambda t$. Prove that $E[M_t \,|\, N_s] = M_s$ for all $t > s \geq 0$. (In fact, M_t is also a martingale, but you don't need to prove the other properties. Observe the general pattern in Problems 4 and 5: one way to obtain a martingale is to subtract the expected value from a stochastic process.)

Problem 6 (20pt) Suppose X_t is a stochastic process satisfying the stochastic differential equation

$$dX_t = b(X_t)dW_t,$$

where

$$b(x) = \begin{cases} 2, & x \geq 0 \\ 1, & x < 0. \end{cases}$$

Use $\Delta t = 0.01$ with 100 time steps to run 1,000 simulations of X_t in R (building on the code from lab to simulate Brownian motion). See page 228 for the simulation formula. Use the simulations to estimate the expected value of X_1 (take the mean value across your simulations) and the probability that $X_1 > 0$ (using the proportion of simulations which have a positive value at $t = 1$). Email your R script to tleise@amherst.edu.