Math 365 Markov Chain Model of Diffusion

The objective of this lab is to explore large-time behavior and equilibria (invariant probability distributions) of finite-state Markov chains. Feel free to discuss problems with each other during lab, as well as ask me questions. Please email your R script with answers included as comments to tleise@amherst.edu.

1 Warm-up problem

Suppose an urn contains 2 balls, where balls can be either blue or red. Initially, both balls are red. At each stage, one ball is removed at random and replaced by a new ball, which with probability 0.8 is the same color as the removed ball and with probability 0.2 is the opposite color.

1. Define X_n to be the number of red balls in the urn after the nth selection. What are the possible states? What is the transition matrix $\mathbf{P}$ for the resulting Markov chain?
2. What is the probability that $X_1 = 2$, given that $X_0 = 2$?
3. What is the probability that $X_5 = 2$, given that $X_0 = 2$? (Use a power of $\mathbf{P}$ to compute this.)
4. What is the probability in the long run that the chain is in state 2? Solve in two ways: (a) raise $\mathbf{P}$ to a large power; (b) compute the left eigenvectors of $\mathbf{P}$ and find the one corresponding to eigenvalue 1.

2 Bernoulli-Laplace model of diffusion

Consider two urns A and B, each of which contains m balls; b of the $2m$ total balls are black and the remaining $2m - b$ are white. At each stage, one ball is randomly chosen from each urn and put in the other urn (done simultaneously, so balls are being swapped between urns). Define X_n to be the number of black balls in urn A after the nth swap. Observe that $X_n = i$ implies A contains i black balls and $m - i$ white balls, while B contains $b - i$ black balls and $m - (b - i)$ white balls. The resulting Markov chain is a probabilistic model of diffusion of two fluids.

Let $m = 4$ and $b = 4$ for purposes of this lab (to keep calculations manageable). Assume urn A initially contains 4 white balls and B contains 4 black balls.

5. What are the possible states? What is the transition matrix $\mathbf{P}$ for the resulting Markov chain? What kind of boundaries does this chain have?
6. What is the probability that $X_2 = 0$, given that $X_0 = 0$?
7. What is the probability that $X_3 = 0$, given that $X_0 = 0$?
8. What is the probability that $X_6 = 0$, given that $X_0 = 0$?

9. Let ϕ_0 be the initial probability distribution corresponding to $X_0 = 0$, and let $\phi_n = \phi_0 \mathbf{P}^n$. As n increases, what is ϕ_n converging to?

   To visualize how the system is evolving, you can run a graphical simulation of the ϕ_n:

```
phi<-c(1,0,0,0,0)
for (k in 1:15) {
phi <- phi %*% P
plot(0:4,phi,ylim=c(0,1))
Sys.sleep(1)
}
```

10. Find the eigenvalues of $\mathbf{P}$.

11. Find the left eigenvector of $\mathbf{P}$ corresponding to eigenvalue 1, and compare to the vector you found in Exercise 9.