Math 365 More Brownian Motion – Last Lab!

Submit your lab work to tleise@amherst.edu (and/or turn in by hand) by next Wednesday.

In this lab we will do some further exploration of Brownian motion.

1 Multi-Dimensional Brownian Motion

Multi-dimensional Brownian motion can be constructed using independent one-dimensional Brownian motions for the coordinates of a vector-valued function $X_t = (X_t^{(1)}, \ldots, X_t^{(n)})$.

Exercise 1 Update the R code from last week's lab to run simulations of 2-dimensional Brownian motion by generating two Brownian motions `x1` and `x2`. Plot the resulting path (`x1`,`x2`) in the xy-plane.

Exercise 2 Add a 3rd Brownian motion `x3` to run simulations of 3-dimensional Brownian motion. Plot the resulting path using `plot3D`, which requires loading the plot3D package: in the RStudio window, click on "Install Packages" in the Packages tab, then choose plot3D. It should then download and install the package, and you will probably need to click on the box next to plot3D in the list of packages to actually load it.

```
lines3D(x1,x2,x3,ticktype="detailed",main="3D Brownian motion simulation")
```

2 Martingales and Brownian Motion

For continuous-time stochastic processes, we say that Y_t is a *martingale* with respect to filtration $\mathcal{F}_t = \{X_s : 0 \leq s \leq t\}$ if it is $\mathcal{F}_t$-measurable, $\mathbb{E}|Y_t| < \infty$, and $E(Y_t | \mathcal{F}_s) = Y_s$ for all $s < t$.

Recall that *geometric Brownian motion* can be defined as

$$Y_t = e^{\mu t + \sigma X_t},$$

where X_t is a standard Brownian motion.

Exercise 3 Calculate $\mathbb{E}[e^{\sigma X_t}] = \int_{-\infty}^{\infty} e^{\sigma x} f(x) dx$, given that X_t is a standard Brownian motion, so $f(x)$ here is the density function for the normal distribution with mean 0 and variance t. As it's not an easy integral to do by hand, feel free to use Mathematica or Wolfram Alpha.

Exercise 4 Calculate $E(Y_t | \mathcal{F}_s)$ for $t > s \geq 0$, using the result from Exercise 3.

Exercise 5 What condition on μ and σ is required for a geometric Brownian motion Y_t to be a martingale?