Oversampling in the Sampling Theorem (Exercises due Friday 11/20)

(Adapted from *A First Course in Wavelets* by Boggess & Narcowich)

Suppose you have a bandlimited function $f(t)$ where the support of $\hat{f}(\xi)$ is in $[-\Omega, \Omega]$ (so $L = 2\Omega$ in the statement of the Sampling Theorem in class). We can perfectly recover $f(t)$ using only the samples $f(\frac{n}{2\Omega})$ for $n \in \mathbb{Z}$. Sampling with spacing greater than $\frac{1}{2\Omega}$ (undersampling) will result in loss of information, but what happens if we oversample, say with spacing $\frac{1}{2a\Omega}$ for some $a > 1$? We want to derive a new formula that optimally makes use of the oversampled values.

Exercise 1 Repeat the first part of the proof of the Shannon Sampling Theorem using $L = 2a\Omega$ to derive the following:

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} c_n e^{\pi i n \xi/(a\Omega)} \quad \text{with} \quad c_n = \frac{1}{2a\Omega} f\left(-\frac{n}{2a\Omega}\right)$$

Exercise 2 Let $\hat{g}_a(\xi)$ be the piecewise linear function with $\hat{g}_a(\xi) = 1$ on $[-\Omega, \Omega]$, $\hat{g}_a(\xi) = 0$ for $|\xi| > a\Omega$, and has linear segments on $[-a\Omega, -\Omega]$ and $[\Omega, a\Omega]$ to generate a continuous function. Show that the inverse Fourier transform is (you may use Mathematica to integrate and simplify if you wish):

$$g_a(t) = \frac{\cos(2\Omega\pi t) - \cos(2a\Omega\pi t)}{2(a-1)\Omega\pi^2 t^2}.$$

Exercise 3 Observe that $\hat{f}(\xi) = 0$ for $|\xi| > \Omega$ and $\hat{g}_a(\xi) = 1$ on $[-\Omega, \Omega]$, so $\hat{f}(\xi) = \hat{f}(\xi)\hat{g}_a(\xi)$ for all ξ. Use this and the first two exercises to rework the rest of the Sampling Theorem proof for the oversampling case $a > 1$ to derive:

$$f(t) = \frac{1}{2a\Omega} \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2a\Omega}\right) g_a\left(t - \frac{n}{2a\Omega}\right)$$

This expression converges faster than the original Shannon sampling formula using sinc functions (need to sum fewer terms to get very good approximation for f because $g_a(t)$ decays like $1/t^2$ while $\text{sinc}(t)$ decays like $1/t$), but you have to take samples more often – there is a trade-off between sample rate and rate of convergence.