Practice Problems for Math 320 Exam 1

The exam will be in class on Monday, October 26, and will cover the material in Chapters 1 and 3-6 (through section 6.4).

Key concepts include:

1. $L^2(\mathbb{T})$ function space and its inner product and norm
2. Properties of complex exponential functions
3. Fourier series for $L^2(\mathbb{T})$ functions
4. Basic results concerning convergence of Fourier series (memorize Dirichlet's theorem and Thm 5.1, but not other specific convergence theorems)
5. Riemann-Lebesgue Lemma (version on p 109) and rate of decay of Fourier coefficients
6. Definition of good kernel
7. Dirichlet kernel and relation to partial Fourier sums
8. Fejér kernel and relation to Cesàro sums
9. Parseval's identity (Fourier series and DFT versions)
10. Convolution (definition and properties for $L^2(\mathbb{T})$ functions)
11. Definition and properties of the discrete Fourier transform
12. Convolution theorem for DFT

Practice problems (exam will have problems very similar to these):

Problem 1 Suppose that $\{g_k(t)\}$ is an orthonormal basis for $L^2(\mathbb{T})$. For $L \in \mathbb{N}$, let $f_L(t) = \sum_{k=-L}^{L} a_k g_k(t)$. State an easy way to compute the a_k coefficients. Show that

$$\|f_L\|^2_{L^2(\mathbb{T})} = \sum_{k=-L}^{L} |a_k|^2.$$

Problem 2 Consider the function $f(t) = e^{-it/2}$ for $-\pi \le t \le \pi$, extended to the real line with period 2π. Find the Fourier series for $f(t)$. Simplify the coefficients as much as possible.

Problem3 Suppose that $f(t)$ and $g(t)$ are even, real-valued functions on $\mathbb{R}$. Prove that the convolution $f * g$ is also an even function.

Problem 4 State the Riemann-Lebesgue Lemma for $f \in L^2(\mathbb{T})$.

Problem 5 Combine integration by parts with the Riemann-Lebesgue Lemma to demonstrate how the number of continuous derivatives a function has relates to the rate at which its Fourier coefficients converge to 0.

Problem 6 State the definition of the Dirichlet kernel, and show why convolution of a function f with the Dirichlet kernel yields a Fourier partial sum for f.

Problem 7 Prove that $f_n = n\chi_{[-\pi/n,\pi/n]}$ for $n \in \mathbb{N}$ is a good kernel.

Problem 8 Carefully state Dirichlet's Theorem for pointwise convergence of Fourier series. Contrast with the situation for L^2-convergence of Fourier series.

Problem 9 State and prove Parseval's identity for the discrete Fourier transform setting.

Problem 10 Let $f(t)$ be defined on the interval $[0, 1)$ as

$$f(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1/2, \\ 0 & \text{if } 1/2 \leq t < 1. \end{cases}$$

Let $\mathbf{x} \in \mathbb{R}^N$ with components $x_m = f(m/N)$ for $0 \leq m \leq N-1$ be a sampled version of $f(t)$ and assume, for simplicity, that N is even. Find the DFT coefficients $\hat{x}_k$ of $\mathbf{x}$ and simplify as much as possible, e.g., $e^{ik\pi} = (-1)^k$, etc.

Problem 11 Let N be a positive even integer and define $\mathbf{x} \in \mathbb{C}^N$ to have components $x(k) = (-1)^k$ for $k = 0, 1, \ldots, N-1$. Find the DFT coefficients $\hat{x}(n)$ and simplify as much as possible. Explain the connection between the number of cycles (repeated pattern of [1 -1]) occurring in $\mathbf{x}$ and its DFT.

Problem 12 Write out the matrix $\mathbf{F}_4$ and verify that multiplying $\mathbf{F}_4$ by its complex conjugate yields 4 times the identity matrix.

Problem 13 State and prove the Convolution Theorem concerning the discrete Fourier transform of a discrete circular convolution of two vectors in $\mathbb{C}^N$.

Problem 14 Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^N$. Show that $y(k) = \overline{x(k)}$ for all $0 \leq k < N$ if and only if $\hat{y}(n) = \overline{\hat{x}(-n)}$ for all $0 \leq n < N$.

Problem 15 Let $\mathbf{f} = [1\ 2\ 3\ 4]^t$ and $\mathbf{g} = [4\ 2\ 8\ 6]^t$. Find a vector $\mathbf{x} \in \mathbb{C}^4$ that satisfies $\mathbf{f} * \mathbf{x} = \mathbf{g}$. Hint: What nice property of convolution would simplify this problem?