Math 320 Linear Spline MRA Homework   Due Wed 12/16

Let $\varphi(x) = 1-|x|$ if $-1 \leq x \leq 1$ and zero otherwise. Let V_j be the space of continuous piecewise-linear functions in $L^2(\mathbb{R})$ with corners only occurring at dyadic points $k/2^j$ for $k \in \mathbf{Z}$.

We can easily show that $\{V_j\}_{j\in\mathbf{Z}}$ satisfies the nested, separation, and scaling properties of a multiresolution analysis, listed in Definition 10.3 on page 265 in the textbook. The density property follows from a standard theorem in real analysis that any $f \in L^2(\mathbb{R})$ can be approximated arbitrarily well by continuous piecewise linear functions. In other words, continuous piecewise linear functions with corners restricted to dyadic points are dense in $L^2(\mathbb{R})$.

1. Verify that $\int_{\mathbb{R}} \varphi(x)dx \neq 0$.
2. Directly check that $\varphi(x)$ and $\varphi(x-1)$ are not orthogonal. Hence $\{\varphi(x-k)\}_{k\in\mathbf{Z}}$ is not an orthogonal set.
3. Find $\hat{\varphi}(\xi)$.
4. Calculate $H(\xi)$.

We need to generate a new piecewise linear scaling function whose integer translates are orthogonal in order to have an MRA. In the exercises below, we will generate a new scaling function that satisfies all of the MRA requirements using transform techniques and, in particular, the stability function, which we define here to be

$$\mathcal{A}(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi+k)|^2 = \sum_{k\in\mathbb{Z}} \langle \varphi(x), \varphi(x-k)\rangle e^{-2\pi ik\xi}.$$

The stability function $\mathcal{A}(\xi)$ has constant value 1 if and only if $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal set.

5. Calculate $\mathcal{A}(\xi)$ for the given $\varphi(x)$ (note that only a finite number of terms in the equivalent Fourier series are nonzero; directly calculate those nonzero terms). Put in simplest form using trig functions rather than complex exponentials. Show that $0 < \frac{1}{3} \leq 2\pi\mathcal{A}(\xi) \leq 1$.

Let $\Phi(\xi) = 1/\sqrt{\mathcal{A}(\xi)}$. We now define a new scaling function φ^b through its Fourier transform:

$$\hat{\varphi^b}(\xi) = \hat{\varphi}(\xi)\Phi(\xi).$$

6. Prove that the scaling function φ^b generates an orthonormal set of translates (check its stability function).

The sequence of approximation spaces $\{V_j\}_{j\in\mathbf{Z}}$ as defined above but with the corrected scaling function φ^b now satisfies all the required properties for an MRA.

7. Find the "symbol" $H(\xi)$ such that $\hat{\varphi^b}(\xi) = H(\xi/2)\hat{\varphi^b}(\xi/2)$. Simplify to an expression involving cosine functions.
8. We define the associated wavelet symbol as $G(\xi) = -e^{-2\pi i\xi}\overline{H(\xi+\frac{1}{2})}$. We can then define the wavelet function $\psi^b(t)$ via $\hat{\psi^b}(\xi) = G(\xi/2)\hat{\varphi^b}(\xi/2)$. Find $G(\xi)$ and simplify as much as possible.