Dido's Problem (Exercises due Wednesday 10/14)

(Adapted from *Fourier Series*, Bhatia)

The epic poem *Aeneid* by Vergil includes this legend:

Fleeing from persecution by her brother, the Phoenician princess Dido set off westward along the Mediterranean shore in search of a haven. A certain spot on the coast of what is now the bay of Tunis caught her fancy. Dido negotiated the sale of land with the local leader, Yarb. She asked for very little–as much as could be "encircled with a bull's hide." Dido managed to persuade Yarb, and a deal was struck. Dido then cut a bull's hide into narrow strips, tied them together, and enclosed a large tract of land. On this land she built a fortress and, near it, the city of Carthage. There she was fated to experience unrequited love and a martyr's death.

–From *Stories about Maxima and Minima*, Tikhomirov

Dido's problem, also called the isoperimetric problem, is to find the simple closed curve that encloses the greatest area, given a fixed value for the perimeter (the length of the tied-together strips of bull's hide). A classical method of solving this and similar problems falls under the heading of calculus of variations. We shall instead make use of Fourier techniques to solve this problem.

Let's represent our simple closed curve C in parametric form: $(x(t), y(t))$ for $t \in [-\pi, \pi]$, and assume that the component functions $x(t)$ and $y(t)$ are continuous and piecewise smooth, satisfy $x(-\pi) = x(\pi)$ and $y(-\pi) = y(\pi)$ (so C is closed), and the point $(x(t), y(t)) \neq (x(s), y(s))$ if $-\pi \leq s < t < \pi$ (so C is simple – doesn't self-intersect). We also wish to fix the arc length of C to equal L:

$$\int_{-\pi}^{\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = L.$$

Exercise 1 With no loss of generality, we may choose the speed of the parametrization (we are only interested in the shape of the trajectory, and not what time each point is hit). Let's assume a constant speed. What must that speed v be? (Recall that speed is given by $v = \sqrt{(x'(t))^2 + (y'(t))^2}$.)

Exercise 2 Integrate both sides of $(x'(t))^2 + (y'(t))^2 = v^2$ over the interval $[-\pi, \pi]]$ (using the v found in Exercise 1) and apply Parseval's Identity (also called Plancherel's Formula):

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta = \sum_{n \in \mathbf{Z}} |\hat{f}(n)|^2,$$

where $\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$ is the Fourier coefficient. Apply the formula for the Fourier coefficient of the derivative f'. Adjust the series to sum from $n = 1$ to ∞ by using the fact that if f is a real-valued function then $\hat{f}(-n)$ is the complex conjugate of $\hat{f}(n)$ (from the definition and fact that $\overline{e^{-i\theta}} = e^{i\theta}$).

Exercise 3 Green's Theorem says that the area enclosed by the curve C is $A = \int_{-\pi}^{\pi} x(t)y'(t)dt$. Use Parseval's Formula:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}d\theta = \sum_{n\in\mathbf{Z}} \hat{f}(n)\overline{\hat{g}(n)}$$

and steps similar to Exercise 2 to rewrite this expression as a series from $n = 1$ to ∞ involving $\hat{x}(n)$ and $\overline{\hat{y}(n)}$.

Exercise 4 Combine the results of Exercises 2 and 3 to find a series that equals $L^2 - 4\pi A$ (don't expect anything beautiful–it's a little messy). Substitute $\hat{x}(n) = \alpha_n + i\beta_n$ and $\hat{y}(n) = \gamma_n + i\delta_n$ into the nth term of this series and simplify. After some factoring, you should find that this expression is the sum of three nonnegative terms $(n^2-1)(\alpha_n^2+\beta_n^2)+(\dots)^2+(\dots)^2$, and so must be nonnegative. This implies that the series must be nonnegative.

Exercise 5 Conclude that $L^2 \geq 4\pi A$. Under what conditions does equality hold? What is the parametrization $(x(t), y(t))$ in this case? What shape does this yield (hint: calculate $(x(t) - \hat{x}(0))^2 + (y(t) - \hat{y}(0))^2$). You have now proven that this shape encloses the largest possible area $(L^2/4\pi)$ bounded by perimeter L.