Fourier Series and the Heat Equation (Due Wed 9/30)

The objective of this lab is to work through a simple example to demonstrate how Fourier series can be used to predict the temperature along a rod over time. The original motivation for the development of Fourier series involved finding solutions to partial differential equations like the heat equation, which describes the diffusion of heat along a bar over time.

As in the previous lab, we will use the sine series as the most convenient formulation of Fourier series for our present purpose.

Suppose we want to follow the diffusion of heat along a bar of length π meters, which we represent mathematically as $[0, \pi]$.

Let $u(x, t)$ be the temperature of the bar at position x (in meters) at time t (in seconds).

At time $t = 0$, the bar has some temperature distribution, say, $u(x, 0) = 100x$ degrees Celsius along the interior of the bar $0 < x < \pi$.

We also need to specify what happens at the ends. Let's say that both ends are held constant at 0 degrees. This implies that $u(0, t) = 0$ and $u(\pi, t) = 0$ for all $t \geq 0$.

Two questions: how does the heat diffuse along the bar over time, and what is the temperature distribution along the bar in the long run?

To answer these questions, we assume that the temperature at each location x of the bar at time t is governed by the heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where α is the rate of diffusion.

Instructions: Submit the requested figures as .fig or .tif files. You can save figures by clicking on File in the figure window, then choosing Save As and naming the file after the corresponding exercise, e.g., JonesLab2Exercise3.fig. When finished the figure files as attachments to tleise@amherst.edu.

Exercise 1 (Separate variables) Set $u(x, t) = f(x)g(t)$. Plug into the heat equation (1) and separate variables, so that functions of t are on the left hand side and functions of x are on the right hand side, putting α with the functions of t. This rearranged equation holds for all x and for all t, so both sides must be equal to some constant we'll call $-\lambda$ (the negative sign makes things work smoothly inside a square root, as we will see below). Now we have two ordinary differential equations (ODEs), one in x and one in t.

Exercise 2 (Solve the ODEs) Verify by taking derivatives that the general solutions to the ODEs are $f(x) = a_1 \cos(\sqrt{\lambda} x) + a_2 \sin(\sqrt{\lambda} x)$ and $g(t) = be^{-\alpha \lambda t}$. From the boundary conditions $u(0, t) = f(0)g(t) = 0$ and $u(\pi, t) = f(\pi)g(t) = 0$ for all t, we see that $f(0) = 0$ and $f(\pi) = 0$. What does this information tell us about the unknown parameter values?

Well, $f(0) = 0$ implies $a_1 = 0$, and $f(\pi) = 0$ then implies that either $a_2 = 0$ (not very interesting) or $\sqrt{\lambda}$ is an integer, call it k. So now we have many possible candidates for the solution, which we can express as $b_k \sin(kx)e^{-\alpha k^2 t}$. We form a sum of these terms, since by the principle of superposition we can add solutions to obtain new solutions to a linear equation.

This solution has form $u(x,t) = \sum_{k=1}^{\infty} b_k \sin(kx)e^{-\alpha k^2 t}$. But what are the values of the b_k? To compute these, we use our final piece of information, that $u(x,0) = \sum_{k=1}^{\infty} b_k \sin(kx) = 100x$. This looks like a sine series! We know how to compute the coefficients for a sine series from the previous lab.

Exercise 3 (Find the sine series) Determine a formula for the b_k coefficients for the function $100x$ on $[0,\pi]$ using the fact that

$$\int x \sin(kx)dx = \frac{\sin(kx)}{k^2} - \frac{x\cos(kx)}{k}.$$

Plot this function and partial sums of the sine series with $n = 10, 20, 50$ on the same figure. Add a legend and axes labels: `xlabel('Position x');ylabel('Temperature u')`.

Exercise 4 (Plot the solution) Let $\alpha = 0.01$ and plot the partial sum of the solution,

$$\sum_{k=1}^{50} b_k \sin(kx)e^{-\alpha k^2 t},$$

at times $t = 1, 10, 50, 100, 200, 400$ together on the same figure. Note that in Matlab, the exponential function e^t is written `exp(t)`. Label the graphs with a legend and add axes labels. What is predicted to happen to the temperature along the bar in the long run?