Averaging and Summability Methods (Due Wed 10/7)

The objective of this lab it to work through examples of the Cesàro and Abel averaging methods to gain a better feel for how they work. The problems are adapted from Section 4.8 of the Pereyra and Ward text.

The main idea is that these two methods have better convergence properties when applied to partial sums of a Fourier series than just taking the limit of partial sums themselves. Before looking at the more complicated setting of Fourier series, we'll start by playing with summing sequences of real numbers and comparing the results of the different averaging methods.

Here the *Cesàro mean* of a sequence $\{b_n\}_{n=0}^{\infty}$ of real numbers is the arithmetic mean

$$\sigma_n := (b_0 + b_1 + b_2 + \cdots + b_{n-1})/n,$$

and the *Abel mean* of $\{b_n\}_{n=0}^{\infty}$ for $0 \leq r < 1$ is defined by

$$A_r := (1-r)\sum_{n=0}^{\infty} r^n b_n.$$

Instructions: Write out your answers to the following questions on paper to submit.

Exercise 1 Suppose the sequence $\{b_n\}_{n=0}^{\infty}$ converges to the real number b, that is, $\lim_{n\to\infty} b_n = b$. In the case of a convergent sequence like this, the limit of the Cesàro means will also be b. The converse is not true however: the convergence of the Cesàro means does not guarantee the convergence of the sequence itself.

(a) What is the Cesàro mean for the sequence with $b_n = 1$ for all n?

(b) What is the Cesàro mean for the sequence with $b_n = 1/2^n$ for all n? Verify that the Cesàro means converge to the same limit as the sequence itself. Hint: use the geometric sum formula.

(c) Find a sequence $\{b_n\}_{n=0}^{\infty}$ such that the Cesàro means converge but the sequence itself does not.

Exercise 2 Given an arbitrary fraction p/q between 0 and 1, find a sequence of ones and zeros such that the Cesàro means converge to p/q. Try some simple examples first like $1/2$ or $2/3$.

Exercise 3 Suppose the sequence $\{b_n\}_{n=0}^{\infty}$ converges to the real number b. In the case of a convergent sequence like this, the Abel mean will also converge to b, but as with the Cesàro mean, the converse does not hold (convergence of the Abel means does not guarantee convergence of the sequence itself).

(a) What is the Abel mean for the sequence with $b_n = 1$ for all n? To what does it converge as $r \to 1^-$?

(b) What is the Abel mean for the sequence with $b_n = (-1)^n$ for all n? To what does it converge as $r \to 1^-$?

Exercise 4 Let $s_n = \sum_{k=0}^{n} b_k$. Prove that, for $0 \leq r < 1$,

$$(1-r)\sum_{n=0}^{\infty} r^n s_n = \sum_{n=0}^{\infty} r^n b_n.$$

Exercise 5 Take the sequence of zeros and ones from Exercise 2 whose Cesàro mean converges to $3/4$ and compute its Abel mean. What is the limit of the Abel mean as $r \to 1^-$?