Math 284 Sol'ns

1.4 #5 (a) Assume $x^* = g(x^*)$ and $g'(x^*) = g''(x^*) = 0$

Then Taylor expansion at x_n about fixed pt x^* is

$$g(x_n) = \cancelto{x^*}{g(x^*)} + \cancelto{0}{g'(x^*)}(x_n - x^*) + \frac{1}{2}\cancelto{0}{g''(x^*)}(x_n - x^*)^2 + \frac{1}{6}g'''(\xi_n^*)(x_n - x^*)^3 \quad \text{for some } \xi_n^* \in [x_n, x^*]$$

$$x_{n+1} = g(x_n) = x^* + \frac{1}{6}g'''(\xi_n^*)(x_n - x^*)^3$$

The ratio of consecutive absolute errors in iteration is then

$$\frac{|x_{n+1} - x^*|}{|x_n - x^*|^3} = \left|\frac{1}{6}g'''(\xi_n^*)\right|$$

As $n \to \infty$, ξ_n^* gets "squeezed" between x_n and x^* to converge to x^* (we assume $x_n \to x^*$ as $n \to \infty$).

Hence $\displaystyle\lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|} = \frac{1}{6}|g'''(x^*)|$

#5 (b) Suppose f has a root of multiplicity m at $x = x^*$:

$$f(x) = (x - x^*)^m h(x), \quad h(x^*) \neq 0$$

We want to examine convergence of Newton's method:

$$g(x) = x - \frac{f(x)}{f'(x)} \quad \text{and} \quad g'(x) = f(x)f''(x)/f'(x)^2$$

Calculate $g'(x)$:

$$g'(x) = \frac{(x-x^*)^m h(x) \cdot [m(m-1)(x-x^*)^{m-2} + 2m(x-x^*)^{m-1}h'(x) + (x-x^*)^m h''(x)]}{[m(x-x^*)^{m-1}h(x) + (x-x^*)^m h'(x)]^2}$$

$$= \frac{m(m-1)h(x)^2 + m(x-x^*)h(x)h'(x) + (x-x^*)^2 h(x)h''(x)}{m^2 h(x)^2 + 2m(x-x^*)h(x)h'(x) + (x-x^*)^2 h'(x)^2}$$

$$g'(x^*) = \frac{m(m-1)h(x)^2 + 0}{m^2 h(x)^2 + 0} = \frac{m-1}{m} = 1 - \frac{1}{m}$$

1^{st} order Taylor expansion is $\cancelto{x_{n+1}}{g(x_n)} = \cancelto{x^*}{g(x^*)}\cancelto{1-1/m}{g'(\xi_n^*)}(x_n - x^*)$ for some ξ_n^* between x_n and x^* (so again $\xi_n^* \to x^*$ as $n \to \infty$)

Ratio of consecutive absolute errors

$$\frac{|x_{n+1} - x^*|}{|x_n - x^*|} = |g'(\xi_n^*)| \to |g'(x^*)| = 1 - \frac{1}{m} \quad \text{as } n \to \infty$$

2.3 #3 Suppose A and B are both unit lower triangular and $n \times n$.

To show $C=AB$ is also unit lower triangular, we need to prove $c_{ii}=1$ for $i=1,\dots,n$ and $c_{ij}=0$ for $i<j$.

We know $a_{ii}=1=b_{ii}$ and $a_{ij}=0=b_{ij}$ for $i<j$.

$$c_{ii}=\sum_{k=1}^{n} a_{ik}b_{ki} = \underbrace{\sum_{k=1}^{i-1} a_{ik}\overset{0}{b_{ki}}}_{\text{since } k<i} + \overset{1}{a_{ii}}\,\overset{1}{b_{ii}} + \underbrace{\sum_{k=i+1}^{n} \overset{0}{a_{ik}}\,b_{ki}}_{\text{since } i<k} = 1$$

$$\text{For } i<j,\quad c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = \underbrace{\sum_{k=1}^{i} a_{ik}\overset{0}{b_{kj}}}_{\text{since } k\le i<j} + \underbrace{\sum_{k=i+1}^{n} \overset{0}{a_{ik}}\,b_{kj}}_{\text{since } i<k} = 0$$

#4 Prove inverse of unit lower triangular is also unit lower tri.

Suppose A is unit lower tri. $n \times n$ matrix.

To find A^{-1}, row reduce $[A \mid I]$ to $[I \mid A^{-1}]$.

(Note we know A is invertible because $\det A = 1$)

Examine $[A \mid I]$:

$$\left[\begin{array}{ccccc|ccc}
1 & 0 & 0 & \cdots & 0 & 1 & & 0 \\
a_{21} & 1 & 0 & & \vdots & & \ddots & \\
a_{31} & a_{32} & 1 & 0 \cdots & 0 & & & \\
\vdots & \vdots & & \ddots & \vdots & 0 & & \\
a_{n1} & a_{n2} & \cdots & a_{n,n-1} & 1 & & & 1
\end{array}\right]$$

To eliminate entries below diagonal in 1^{st} column, subtract a_{k1} times 1^{st} row from k^{th} row. On right hand side, this can only change entries below diagonal.

Continue by eliminating entries below diagonal in 2^{nd} column using multiples of new 2^{nd} row (note a_{21} has been changed to 0 by first step). Again, RHS is only changed below the diagonal.

Repeat through column $(n-1)$, so LHS is identity matrix and RHS is a unit lower triangular matrix that is A^{-1}.

Alternatively, let $B=A^{-1}$. For $i<j$, $0=\sum_{k=1}^{n} a_{ik}b_{kj} = b_{ij}+\sum_{k=1}^{i-1} a_{ik}b_{kj}$

since $AB=I$ and A is unit lower triangular.

Letting $i=1$ yields $b_{1j}=0$ for all $j>1$. Once we know $b_{mj}=0$ for $j>m$, the eq. about yields $b_{(m+1)j}=0$ for all $j>m+1$. By induction, we have $b_{ij}=0$ for all $i<j$. For $i=j$, $1=\sum_{k=1}^{n} a_{ik}b_{ki} = a_{ii}b_{ii} = b_{ii}$.

Hence $B=A^{-1}$ is unit lower triangular.

2.4 #9 (a) Suppose A is a symmetric matrix. By the Spectral Theorem, A's eigenvalues are real-valued and there is an orthonormal basis of eigenvectors $\{\vec{u}_1, \cdots, \vec{u}_n\}$. Let $U = [\vec{u}_1 \cdots \vec{u}_n]$ and D be the diagonal matrix of the corresponding eigenvalues λ_k. Then $A = UDU^t$

First suppose A is positive definite, so $\vec{x}^t A \vec{x} > 0$ for all $\vec{x} \neq \vec{0}$. Use the eigenvectors: $A\vec{u}_k = \lambda_k \vec{u}_k$ (note $\vec{u}_k \neq \vec{0}$)
This leads to $\vec{u}_k^t A \vec{u}_k = \vec{u}_k^t(\lambda_k \vec{u}_k) = \lambda_k \|\vec{u}_k\|^2 = \lambda_k$,
hence $\lambda_k > 0$, for $k = 1, 2, \cdots, n$.

Now suppose the eigenvalues are all positive: $\lambda_k = 0 \ \forall k$
Take any $\vec{x} \neq 0$ and define $\vec{y} = U^t \vec{x}$, which is also not the zero vector since U^t is an invertible matrix.
Examine $\vec{x}^t A \vec{x} = \vec{x}^t UDU^t \vec{x} = \vec{y}^t D \vec{y} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$
The eigenvalues are all positive and not all y_k are zero, so this sum must be positive: $\vec{x}^t A \vec{x} > 0$.

2.5 #3 (a) Suppose $\|A\| = 1$. Note that the definition of $\lim\limits_{n\to\infty} A_n = B$ for a sequence $\{A_n\}$ of matrices is that for all $\varepsilon > 0$ $\exists N$ such that $\|A_n - B\| < \varepsilon$ for all $n \geq N$.
(Replace abs value in regular def'n of limit for numbers with matrix norm.) So we need to show $\|A^n - 0\| \to 0$ as $n \to \infty$, that is, $\lim\limits_{n\to\infty} \|A^n\| = 0$.

$0 \leq \|A^n\| \leq \|A\|^n$ and $\lim\limits_{n\to\infty} \|A\|^n = 0$ (since $\|A\| < 1$)

By Squeeze Theorem, $\lim\limits_{n\to\infty} \|A^n\| = 0$, so $\lim\limits_{n\to\infty} A^n = 0$

scalar zero

zero matrix