Spring 2014 Math 272 Final Exam Review Sheet

You will not be allowed use of a calculator or any other device other than your pencil or pen and some scratch paper. Notes are also not allowed. In kindness to your fellow test-takers, please turn off all cell phones and anything else that might beep or be a distraction.

Be able to calculate:

- Row-reduced echelon form (REF) of a coefficient or augmented matrix
- General solution to a linear system of equations (particular solution plus general solution to the homogeneous system)
- Rank, nullity, null space, column space, and row space of a matrix
- Kernel and range of a linear transformation (including those involving polynomial and matrix spaces), expressed as the span of a basis
- Inverse of 2x2 and 3x3 matrix
- Whether a set of vectors is linearly independent
- Dimension of a vector space
- Whether a linear transformation is one-to-one, onto, or invertible
- Matrix representation of a linear map, given bases for the domain and codomain
- Eigenvalues and eigenvectors of 2x2 and 3x3 matrices (real or complex)
- **D** and **P** matrices for diagonalization of a square matrix **A**; $\mathbf{A}^k$ using **D** and **P**
- Determinant of 2x2 and 3x3 matrices
- Orthonormal basis using Gram-Schmidt procedure
- Projection of a vector onto a subspace
- Orthogonal complement $W^{\perp}$ of a subspace W of a vector space
- Least squares solution to an inconsistent system (using normal equations)
- Singular value decomposition of a 2x2, 3x2, or 2x3 matrix

Know well and understand:

- Leading ones in the REF of a matrix and how to interpret them
- Definitions of linear algebra terms including linearly independent and dependent, span, linear combination, nonsingular, invertible, one-to-one, onto, linear transformation, vector space, subspace, null space, column space, basis, dimension, rank, eigenvalue, eigenvector, kernel, range, similar matrices, orthogonal matrices, orthonormal basis, symmetric matrices, inner product
- Properties of invertible matrices (e.g., columns form a linearly independent set, determinant is nonzero, the matrix represents a linear transformation that is one-to-one, and so on)
- Rank/Nullity Theorem: rank(**A**)+nullity(**A**) equals the number of columns of **A**, and the equivalent formula for linear transformations (dimension of the range plus the dimension of the kernel equals the dimension of the domain)
- Matrix multiplication, distributive rules for transpose and inverse of a matrix product
- Matrix representation of a linear transformation
- Inner products and orthogonal vectors; least squares application

- Properties of determinant
- Principal Axes Theorem: symmetric matrices have real eigenvalues and orthogonal eigenvectors

Start by seeing how many of these things you can do without looking them up. Focus attention on those items that give you trouble. Be sure to work lots of problems, including relevant homework problems, review exercises, and exam problems. Finally, work the practice exams like they're real: don't look anything up until after you have done as much as you can. This will reveal the areas where you need to spend the most time studying.

For further practice beyond this set of review problems, work through the Practice Proofs sheet posted on the course webpage (next to Exam 3) and the 2013 Math 272 final exam.

Some practice problems

1. Prove that $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ is a basis for $\mathbf{R}^3$. Find the coordinates of $\begin{bmatrix}2\\3\\4\end{bmatrix}$ with respect to this basis.
2. Let V={X: AX=XA}, where A=$\begin{bmatrix}1&0&0\\0&1&0\\3&1&2\end{bmatrix}$. Prove that V is a subspace of $\mathbf{M}_{33}$. Find a basis for V and the dimension of V.
3. Show that for a linear transformation T:V→V, ker(T)⊆ker(T∘T). (T∘T means T composed with T).
4. Let W=span$\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix},\begin{bmatrix}4\\2\\2\\1\end{bmatrix}\right\}$. Find an orthonormal basis for $W^\perp$.
5. Find bases for the kernel and range of the transformation $T: P_2 \to P_3$ defined by $T(p(x)) = p(x) + 3xp(x)$. Prove that T is linear. Is it one-to-one? Is it onto? What is the matrix representation of this transformation with respect to the bases $\{1, x, x^2\}$ and $\{1, x, x^2, x^3\}$?
6. Prove that the null spaces of **A** and $\mathbf{A}^t\mathbf{A}$ are identical for any mxn matrix **A**.
7. Rewrite the system $\begin{aligned}x - y + 2z - w &= -1\\ 2x + y - 2z - 2w &= -2\\ -x + 2y - 4z + w &= 1\\ 3x - 3w &= -3\end{aligned}$ as a matrix equation **Ax**=**b**. Solve this system using Gaussian elimination. Find bases for the column and null space of the matrix **A**.

8. Suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthogonal basis for $\mathbf{R}^n$ where each $\mathbf{u}_k$ is a unit vector and let $\mathbf{A}$ be the nxn matrix defined by $\mathbf{A}=c_1 \mathbf{u}_1 \mathbf{u}_1^T+\ldots+ c_n \mathbf{u}_n \mathbf{u}_n^T$ for some scalars $c_1, \ldots, c_n$. Prove that $\mathbf{A}$ is symmetric and has eigenvalue c_k with eigenvector $\mathbf{u}_k$ for $k=1,\ldots,n$.

9. State whether each of the following is true or false. If it is true, briefly explain why it is true. If it is false, then give a true statement and briefly explain why the original was incorrect and why the new statement is correct.
   a. Asking whether the linear system corresponding to an augmented matrix $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{b}]$ is consistent is equivalent to asking whether $\mathbf{b}$ is in the span of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.
   b. $\mathbf{R}^2$ is a subspace of $\mathbf{R}^3$.
   c. A linearly independent set in a vector space V is a basis for V.
   d. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly independent set, then so is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

10. Prove that rank($\mathbf{A}$+$\mathbf{B}$) $\leq$ rank$\mathbf{A}$ + rank$\mathbf{B}$.

11. Suppose that $\mathbf{A}$ is an mxn matrix such that $\mathbf{Ax}=\mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbf{R}^m$. Prove that $\mathbf{A}$ must in fact be square and nonsingular.

12. Find the line that best fits the points (0,0), (1,-1), and (3,-4), in the least squares sense.

13. Find the singular value decomposition of $A=\begin{bmatrix} -2 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$.

14. Find the singular value decomposition of $A=\begin{bmatrix} \sqrt{3} & 2 \\ 0 & \sqrt{3} \end{bmatrix}$.

Partial solutions:

1. The matrix whose columns are these 3 vectors has nonzero determinant, so the set is a basis. The coordinates are [3, -1, 0].
2. Show that V contains the zero polynomial and is closed under scalar multiplication and vector addition. A basis is $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix} \right\}$
3. Take any v in ker(T). $T(v)=0_V$ (zero vector in V), so $T\circ T\,(v)=T(T(v))=T(0_V)=0_V$ (because T is linear). Therefore v is in ker($T\circ T$). We conclude that $\ker(T)\subseteq\ker(T\circ T)$.
4. $\text{Span}\left\{ \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{42} \\ 1/\sqrt{42} \\ -2/\sqrt{42} \\ 6/\sqrt{42} \end{bmatrix} \right\}$

5. Kernel is $\{0\}$; range is spanned by $\{1+3x, x+3x^2, x^2+3x^3\}$. The transformation is one-to-one but not onto (you should prove each of these). The matrix representation is $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.
6. Idea behind proof: If $\mathbf{Ax}=\mathbf{0}$ then clearly $\mathbf{A}^t\mathbf{Ax}=\mathbf{0}$. If $\mathbf{A}^t\mathbf{Ax}=\mathbf{0}$ then $\|\mathbf{Ax}\|^2=(\mathbf{Ax})\cdot(\mathbf{Ax})=\mathbf{x}^t\mathbf{A}^t\mathbf{Ax}=0$ and so $\mathbf{Ax}=\mathbf{0}$ (the only vector with length zero is the zero vector). Remember that a full statement of the proof must show that each set is a subset of the other set, and to prove that, take an arbitrary element from one set and prove it is also in the other set.
7. General solution is given by $x=-1+s$, $y=2r$, $z=r$, $w=s$.
8. Check that the transpose of $\mathbf{A}$ equals itself and that $\mathbf{Au}_k=c_k\mathbf{u}_k$.
9. a. True, since the column space of $\mathbf{A}$ is the set of all vectors $\mathbf{b}$ for which $\mathbf{Ax}=\mathbf{b}$ is consistent (has at least one solution).
   b. False, $\mathbf{R}^2$ is NOT a subspace of $\mathbf{R}^3$. A set like $\{[x,y,0]: x,y \text{ real}\}$ is a two-dimensional subspace of $\mathbf{R}^3$. (This can be viewed as an embedding of $\mathbf{R}^2$ into $\mathbf{R}^3$.) One *can* say that $\mathbf{R}^2$ is isomorphic to a subspace of $\mathbf{R}^3$.
   c. False, because we don't know whether the set spans V. A set of linearly independent vectors that spans V is a basis.
   d. True, since if a vector in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ can be written as a combination of the other two, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ must be a linearly dependent set.
10. The column space of $\mathbf{A}+\mathbf{B}$ is a subspace of the sum (in the set sense) of the column spaces of $\mathbf{A}$ and $\mathbf{B}$, since $(\mathbf{A}+\mathbf{B})\mathbf{x}=\mathbf{Ax}+\mathbf{Bx}\in \text{Col}\mathbf{A}+\text{Col}\mathbf{B}$. The dimension of a subset must be less than or equal to the dimension of the set itself, so we have the following: $\text{rank}(\mathbf{A}+\mathbf{B})=\dim(\text{Col}(\mathbf{A}+\mathbf{B})) \le \dim(\text{Col}\mathbf{A}+\text{Col}\mathbf{B}) \le \dim(\text{Col}\mathbf{A})+\dim(\text{Col}\mathbf{B})=\text{rank}\mathbf{A}+\text{rank}\mathbf{B}$. We used the fact that $\dim(U+V) \le \dim U+\dim V$ for subspaces U and V in the second inequality.
11. The REF of $\mathbf{A}$ must have a leading 1 in every column, otherwise there would be a free variable. Also, the RREF must have a leading 1 in every row, otherwise the system would be inconsistent for some $\mathbf{b}$. Hence A is square and nonsingular.
12. Solve the normal equation to obtain $y=-\frac{19}{14}x+\frac{1}{7}$.
13. $V=I_2$, $U=\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $\Sigma=\begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$
14. $U=\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$, $V=\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$, $\Sigma=\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

Practice Final Exam

1. Consider the matrix $\mathbf{A}=\begin{bmatrix}1&2&3&4\\1&0&1&0\\2&4&5&6\end{bmatrix}$.
   (a) Find a basis for the column space of **A**.
   (b) Find a basis for the null space of **A**.
   (c) Is $\mathbf{b}=[2\ 2\ 4]^T$ in the column space of **A**? If so, write down the general solution of **Ax**=**b** as the sum of a particular solution and the general solution to the homogeneous system.

2. Suppose an $n\times n$ matrix **A** satisfies $\mathbf{A}^2=\mathbf{A}$. What are all of the possible eigenvalues of **A**? To what fundamental space (e.g., column or null space of **A**) does the eigenspace for each eigenvalue of **A** correspond?

3. Consider each matrix below. If it is diagonalizable, find a diagonal matrix **D** similar to **A** and use this information to find a formula for $\mathbf{A}^k$, where k is a positive integer. Simplify your formula as much as possible.

   (a) $\mathbf{A}=\begin{bmatrix}2&1\\1&2\end{bmatrix}$ (b) $\mathbf{A}=\begin{bmatrix}-3&1\\-1&-1\end{bmatrix}$

4. Prove that the transformation $T:P_2\to R^3$ defined by $T(p)=\begin{bmatrix}p(1)\\p'(1)\\p''(1)\end{bmatrix}$ is linear, one-to-one, and onto. Find its matrix representation $\left[T\right]_{B,C}$ relative to bases $B=\{2+x, 1+2x^2, x+x^2\}$ and $C=\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\-1\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

5. Show that $W=\{p(x)\in P_2:\ p(1)=0\}$ is a subspace of P_2. Find a basis for W and $\dim W$.

6. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for a vector space V. Prove that $\{\mathbf{v}_1, \mathbf{v}_1+\mathbf{v}_2, \mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3\}$ is also a basis for V.

7. Suppose **A** is an mxn matrix with $m<n$. Can the null space of **A** contain only the zero vector? Can the column space of **A** be all of $\mathbf{R}^m$? What if $n<m$?

8. State whether each of the following is true or false. If it is true, **briefly** explain why it is true. If it is false, then give a true statement and **briefly** explain why the original was incorrect and why the new statement is correct.
   (a) If **A** is an $m\times n$ matrix whose columns do not span $\mathbf{R}^m$, then the equation **Ax**=**b** is inconsistent for some **b** in $\mathbf{R}^m$.
   (b) The homogeneous equation **Ax**=**0** has the trivial solution if and only if this equation has at least one free variable.

(c) A linear transformation $T:\mathbf{R}^n \to \mathbf{R}^m$ is completely determined by its effect on the columns of the $n\times n$ identity matrix.
(d) If the columns of an $m\times n$ matrix **A** are linearly independent, then the columns of **A** span $\mathbf{R}^n$.

9. What must be true about s and t if the matrix $\mathbf{A} = \begin{bmatrix} 2 & 2 & s \\ 2 & 3 & t \\ 4 & 5 & 7 \end{bmatrix}$ is **not** invertible?
10. Prove that for every mxn matrix **A** with real entries, the equation $\mathbf{A}^t\mathbf{A}\mathbf{x}=\mathbf{A}^t\mathbf{b}$ is consistent for all vectors **b** in $\mathbf{R}^m$.

<u>Solutions:</u>

1. **A** row reduces to $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ so the column space is all of $\mathbf{R}^3$ since the three pivots indicate three linearly independent columns. The column space is therefore spanned by any three linearly independent vectors, such as the first three columns of **A**. The null space is the span of $\{[2\ 0\ {-2}\ 1]^t\}$, and since $\mathbf{b}=[2\ 2\ 4]^t$ is two times the first column, the general solution to $\mathbf{Ax}=\mathbf{b}$ is $\mathbf{x}=[2\ 0\ 0\ 0]^t + t[2\ 0\ {-2}\ 1]^t$.
2. Suppose $\mathbf{Ax}=\lambda\mathbf{x}$ and $\mathbf{A}^2=\mathbf{A}$. Then we must have $\lambda^2=\lambda$, so that either $\lambda=0$ or $\lambda=1$. The eigenspace of $\lambda=0$ corresponds to the null space of **A**, since both the null space and this eigenspace are the set of all vectors **x** that satisfy $\mathbf{Ax}=\mathbf{0}$. If **x** is an eigenvector for $\lambda=1$, then $\mathbf{x}=\mathbf{Ax}$ is in the column space. We must also check that every vector in the column space is an eigenvector. Since $\mathbf{A}^2=\mathbf{A}$, that is, $\mathbf{A}[\mathbf{a}_1 \dots \mathbf{a}_n]= [\mathrm{A}\mathbf{a}_1 \dots \mathrm{A}\mathbf{a}_n]= [\mathbf{a}_1 \dots \mathbf{a}_n]$, each column of **A** is an eigenvector for $\lambda=1$. Therefore the eigenspace for $\lambda=1$ is exactly the column space.
3. (a) $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ (b) Not diagonalizable.
4. This transformation has matrix representation $\begin{bmatrix} 3 & 3 & 2 \\ 1 & 4 & 3 \\ -2 & 5 & 3 \end{bmatrix}$, which has nonzero determinant, so the transformation is linear, one-to-one, and onto (so is invertible).
5. Show that W contains the zero polynomial and is closed under scalar multiplication and vector addition. A basis for W is $\{x-1, x^2-1\}$ and $\dim W=2$.
6. Show that the set is linearly independent (the only linear combination of the three vectors that equals the zero vector has all coefficients equal to 0). Since it has the same number of vectors as a known basis of V, it must be also be a basis.
7. If $m<n$, then the null space cannot be $\{\mathbf{0}\}$, but we can't rule out that the column space may be all of $\mathbf{R}^m$. ($\mathrm{rank}\mathbf{A}\le m<n$, so by the Rank Theorem, $\mathrm{null}\mathbf{A}=n-\mathrm{rank}\mathbf{A}>n-m>0$.) If $n<m$, the null space could possibly be $\{\mathbf{0}\}$, but the column space cannot be all of $\mathbf{R}^m$ since $\mathrm{rank}\mathbf{A}\le n<m$.
8. (a) True; (b) False, a matrix has trivial null space if and only if there are no free variables; (c) True, since the columns of the identity matrix form a basis of the domain of this transformation; (d) False, this is only true if $m=n$. Also note that the n linearly

independent columns lie in $\mathbf{R}^m$, so they form a basis for an n-dimensional subspace of $\mathbf{R}^m$.

9. $s+t=7$
10. By practice exercise 6, null$\mathbf{A}$=null$\mathbf{A}^t\mathbf{A}$. Both $\mathbf{A}$ and $\mathbf{A}^t\mathbf{A}$ have n columns, so by the Rank/Nullity Theorem, they have the same rank. The normal equation is consistent if we can show rank($\mathbf{A}^t\mathbf{A}$)=rank[$\mathbf{A}^t\mathbf{A}$ $\mathbf{A}^t\mathbf{b}$] (augmented matrix). [$\mathbf{A}^t\mathbf{A}$ $\mathbf{A}^t\mathbf{b}$]= $\mathbf{A}^t$[$\mathbf{A}$ $\mathbf{b}$], so rank($\mathbf{A}^t\mathbf{A}$)≤ rank[$\mathbf{A}^t\mathbf{A}$ $\mathbf{A}^t\mathbf{b}$]= rank($\mathbf{A}^t$ [$\mathbf{A}$ $\mathbf{b}$])≤ rank($\mathbf{A}^t$)= rank($\mathbf{A}^t\mathbf{A}$). Here we used that adding a column will either leave the rank the same or increase it and that the rank of a product is less than or equal to the rank of the first matrix in the product. Since the first and last expressions are the same, all of the ranks must be equal in this inequality. In particular, we have rank($\mathbf{A}^t\mathbf{A}$)=rank[$\mathbf{A}^t\mathbf{A}$ $\mathbf{A}^T\mathbf{b}$], so the normal equation is consistent.