Spring 2014 Math 272 Exam 3 Review Sheet

The exam will be in class on Wednesday, April 30, covering Chapters 4 (focusing on 4.8-4.10) and 5. You will not be allowed use of a calculator or any other device other than your pencil or pen and some scratch paper. Notes are also not allowed. In kindness to your fellow test-takers, please turn off all cell phones and anything else that might beep or be a distraction.

Important topics:

- Null space, column space, and row space of a matrix
- Kernel and range of a linear transformation
- Rank and nullity of matrices and linear transformations
- Finding a basis for a given subspace or for the null space, column space, or row space of a matrix A or for the kernel or range of a linear transformation
- Coordinates with respect to a basis
- Change of basis and matrix representation of a linear operator
- Diagonalization and determining whether a matrix is diagonalizable

If you have problems with certain topics or want more practice, feel free to stop by my office.

<u>Review problems:</u>
Chapter 4 Review (p316): #40- 46
Chapter 5 Review (p374): #1-5, 9, 12

<u>Review problems</u>

1. Determine the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}=\mathbf{PDP}^{-1}$.

2. Give an example of a matrix that is not diagonalizable.

3. Let $\mathbf{A}$ be a 2x2 matrix with eigensystem $\left\{\left\{\lambda_1 = -\frac{1}{2}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}, \left\{\lambda_2 = 1, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}\right\}$.

   Find a formula for $\mathbf{A}^k$.

4. Under what conditions on a, b, and c will the matrix $\begin{bmatrix} 2 & 0 & 0 \\ a & 2 & 0 \\ b & c & -1 \end{bmatrix}$ be diagonalizable? To what diagonal matrix will it be similar?

5. Prove that if $\mathbf{A}$ is diagonalizable such that all of its eigenvalues are either 0 or 1, then $\mathbf{A}$ is idempotent, $\mathbf{A}^2=\mathbf{A}$.

6. Let A be an mxn matrix. (a) What must the dimension of the null space of **A** plus the dimension of the column space of **A** equal? (b) What must the dimension of the null space of $\mathbf{A}^t$ plus the dimension of the column space of $\mathbf{A}^t$ equal?

7. Find the column space, null space, and row space of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$.

   Show that the rank/nullity theorem holds for this matrix.

8. Suppose **A** is an mxn matrix and **B** is an nxp matrix. Prove that null(**B**)⊆null(**AB**) and col(**AB**)⊆col(**A**). Under what condition will equality hold in each statement?

9. Find the kernel and range of the linear operator T: $\mathbf{R}^2 \rightarrow P_2$ defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (a-b)+(a+b)x+(a+b)x^2$. Find the rank and nullity of T. Is T one-to-one? onto?

10. Find the kernel and range of the linear operator T: $\mathbf{R}^{2,2} \rightarrow \mathbf{R}^3$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b+c \\ d \end{bmatrix}$. Find the rank and nullity of T. Is T one-to-one? onto?

11. Let the linear transformation T: $\mathbf{R}^2 \rightarrow P_2$ defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (a-b)+(a+2b)x+(a+3b)x^2$. Find the matrix representation of this transformation relative to the standard unit basis for $\mathbf{R}^2$ and the basis $\{1, x, x^2\}$ for P_2.

12. Let $T : \mathbf{P}_2 \rightarrow \mathbf{M}_{22}$ be the transformation that maps a polynomial $p(x)$ to the matrix $\begin{bmatrix} p(0) & p(1) \\ p'(0) & p'(1) \end{bmatrix}$, where $p'(x)$ is the derivative of $p(x)$. Find the kernel and range of T. Is T one-to-one? Is T onto? Find the matrix representation of T relative to the basis $\{x^2, x, 1\}$ for P_2 and $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ for $\mathbf{M}_{22}$.

13. Find the matrix representation of the linear transformation T: $\mathbf{P}_2 \rightarrow \mathbf{R}^3$ defined via $T(p) = \begin{bmatrix} p(0) \\ p(1) \\ p(0) \end{bmatrix}$ using the basis $\{1, 1+2x, 3x+x^2\}$ for P_2 and the standard unit basis for $\mathbf{R}^3$.

Partial solutions

1. $\mathbf{D}=\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$, $\mathbf{P}=\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$
2. For example, $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$, which has a repeated eigenvalue but the associated eigenspace has dimension 1.
3. $\frac{1}{3}\begin{bmatrix} 2+(-1/2)^k & 2+(-1/2)^{k-1} \\ 1-(-1/2)^k & 1-(-1/2)^{k-1} \end{bmatrix}$
4. Need a=0 to be diagonalizable, in which case **A** is similar to diagonal matrix **D** with diagonal entries 2,2,-1.
5. Suppose **A** is diagonalizable such that all of its eigenvalues are either 0 or 1. Then $\mathbf{A}=\mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, where **D** is a diagonal matrix with zeros and ones along the diagonal and **P** is an invertible matrix whose columns are the corresponding eigenvectors. Note that $\mathbf{D}^2=\mathbf{D}$ because $0^2=0$ and $1^2=1$. We have $\mathbf{A}^2=\mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{P}\mathbf{D}\mathbf{P}^{-1}=\mathbf{P}\mathbf{D}^2\mathbf{P}^{-1}=\mathbf{P}\mathbf{D}\mathbf{P}^{-1}=\mathbf{A}$.
6. (a) The number of columns, n. (b) m.
7. The column space is $\mathbf{R}^4$. $\text{null}(\mathbf{A})=\text{Span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$ $\text{row}(\mathbf{A})=\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix},\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$
8. Suppose **A** is an mxn matrix and **B** is an nxp matrix. Let **x** be any vector in null(**B**), so **Bx**=**0**. Multiply each side by **A** to obtain **ABx**=**A0**=**0**, implying that **x** is in null(**AB**) and so null(**B**)⊆ null(**AB**). Let **y** be any vector in col(**AB**), so **y**=**ABx** for some vector **x**. Then **y**=**A(Bx)**=**Az**, where **z**=**Bx**, and so **y** is in the column space of **A**. Therefore col(**AB**)⊆ col(**A**). If **A** is invertible, then null(**B**)=null(**AB**). If **B** is invertible, then col(**AB**)=col(**A**) (you should prove these).
9. T is one-to-one but not onto. The kernel is $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$ and the range is $\text{Span}\left\{1+x+x^2,-1+x+x^2\right\}$.
10. T is onto but not one-to-one. The kernel is $\text{Span}\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}$ and the range is $\mathbf{R}^3$.
11. $\begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$
12. T is one-to-one but not onto. The kernel is {0} and the range is $\text{Span}\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\right\}$. Matrix rep is $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 4 \\ 1 & 1 & 0 \end{bmatrix}$

Please let me know if you find any errors in these solutions. Thanks!