# Spring 2014 Math 272 Exam 1 Review Sheet

The exam will be in class on Monday, February 24.

You will not be allowed use of a calculator or any other device other than your pencil or pen and some scratch paper. Notes are also not allowed. In kindness to your fellow test-takers, please turn off all cell phones and anything else that might beep or be a distraction.

Important topics:

- Row reduction to find the REF of a coefficient or augmented matrix
- Finding the general solution and the solution set of a linear system of equations: unique solution, infinitely many solutions (involving free variables), or no solution
- Matrix addition and multiplication; matrix inverses
- Vector space $\mathbf{R}^n$ and subspaces of $\mathbf{R}^n$
- Linear independence, span, basis, and dimension
- Dot product and norm in $\mathbf{R}^n$
- Elementary matrices
- Transpose and conjugate transpose
- Symmetric matrices
- Matrix transformations

Review problems:
Chapter 1 Review (p62): #7a,8a,14,15,16,17,18,19,21,24
Chapter 2 Review (p167): #2,4ab,7,10,11,13ab,14,17,19,22,23

1. Consider the following system where $c$ is an unknown number:
$$\begin{aligned}(c-2)x+3y&=0\\ 2x+(c-3)y&=0\end{aligned}$$
   (a) Determine all values of $c$ for which the system is consistent.
   (b) Determine all values of $c$ for which the system has a unique solution.
   (c) Determine all values of $c$ for which the system has infinitely many solutions, and state the general solution.

2. Let **A** be a 4x3 matrix and suppose **Ax**=**b** has a unique solution for some vector **b**. Is it possible that there are infinitely many solutions to **Ax**=**c** for some vector **c**? What if **A** were 3x4 instead?

3. Consider the following system where $c$ is an unknown number:
$$\begin{aligned}2x+2y+3z&=0\\ 4x+8y+12z&=-4\\ 6x+2y+cz&=4\end{aligned}$$
   a. Determine all values of $c$ for which the system is consistent.
   b. Determine all values of $c$ for which the system has a unique solution.
   c. Determine all values of $c$ for which the system has infinitely many solutions, and state the general solution.

4. Find a condition on the $b$'s that determines whether the following system is consistent, and state the general solution for the case that this condition holds:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= b_1 \\ x_1 + 4x_2 + 2x_3 &= b_2 \\ 2x_1 + x_2 - 3x_3 &= b_3 \end{aligned}$$

5. Find a 2x3 matrix **A** such that the solution to the homogeneous system **Ax**=**0** is **x**=$x_3 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

6. Suppose **A** is a 3x3 matrix and **y** is a vector in $\mathbf{R}^3$ such that the equation **Ax**=**y** does not have a solution. Does there exist a vector **z** in $\mathbf{R}^3$ such that the equation **A**x=**z** has a unique solution? Fully explain your answer.

7. Let **C** be a $3 \times 5$ matrix. If a solution to **Cx=b** exists, can it be unique? Explain.

8. Give an example of each of the following or explain in one or two sentences why it would be impossible to do so.

   (a) A linear system of three equations in three unknowns that has all nonzero coefficients and no solutions.

   (b) A homogeneous linear system with no solutions.

   (c) A linear system of three equations in five unknowns for which the REF of the augmented matrix has 3 leading ones.

   (d) A linear system of five equations in three unknowns for which the REF of the augmented matrix has 5 leading ones.

9. Let **A** be a square matrix and $n$ a positive integer. Use induction to prove that $(\mathbf{A}^n)^t=(\mathbf{A}^t)^n$.

10. Assume that **A** is a square matrix that satisfies $\mathbf{A}^2$-3**A**+**I**=0. Show that **A** is invertible and $\mathbf{A}^{-1}$=3**I**-**A**.

11. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$.

    (a) Find elementary matrices $\mathbf{E}_1$ and $\mathbf{E}_2$ such that $\mathbf{E}_2\mathbf{E}_1\mathbf{A}$=I.
    (b) Write $\mathbf{A}^{-1}$as a product of elementary matrices.
    (c) Write **A** is as a product of elementary matrices.

12. Find a matrix **A** such that $\mathbf{A}\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{A}\begin{bmatrix}2\\7\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}$. What is the span of the columns of **A**? Are the columns linearly independent?
13. Find a matrix **A** such that $\mathbf{A}\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{A}\begin{bmatrix}2\\7\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix}$. What is the span of the columns of **A**? Are the columns linearly independent?

<u>Partial solutions:</u>

1. Consistent for all c; infinitely many solutions iff c=0 or 5; otherwise unique solution.
2. The REF of A will have a leading 1 in every column, so the solution must be unique for any RHS vector. If **A** is 3x4, we must have infinitely many solutions if the system is consistent (it's not possible to have a leading 1 in every column of a 3x4 matrix).
3. Consistent for all values of $c$, unique solution iff $c \neq 3$, infinitely many solutions iff $c=3$.
4. Condition is $b_3=7b_1-5b_2$.
5. Check your answer by substituting the given solution into the system for your **A**.
6. No,…think about how many leading ones there can be.
7. No, there must be at least 2 free variables in the general solution.
8. (b) and (d) are impossible, but (a) and (c) are possible
9. Base case ($n$=1): $(\mathbf{A})^t=\mathbf{A}^t$. Inductive step: Suppose $(\mathbf{A}^n)^t=(\mathbf{A}^t)^n$ is true for some positive integer $n$. We need to prove it's true for n+1. Using the base case and the inductive step assumption, $(\mathbf{A}^{n+1})^t=(\mathbf{A}\mathbf{A}^n)^t=(\mathbf{A}^n)^t\mathbf{A}^t=(\mathbf{A}^t)^n\mathbf{A}^t=(\mathbf{A}^t)^{n+1}$. By induction, the statement is therefore true for all positive integers $n$.
10. Let **B**=3**I**-**A** and prove that **AB**=**I**=**BA**.
11. $\mathbf{E}_1=\begin{bmatrix}1 & 0\\-3 & 1\end{bmatrix}, \mathbf{E}_2=\begin{bmatrix}1 & 0\\0 & 1/4\end{bmatrix}, \mathbf{E}_1^{-1}=\begin{bmatrix}1 & 0\\3 & 1\end{bmatrix}, \mathbf{E}_2^{-1}=\begin{bmatrix}1 & 0\\0 & 4\end{bmatrix}$.

    $\mathbf{A}=\mathbf{E}_1^{-1}\mathbf{E}_2^{-1}$, $\mathbf{A}^{-1}=\mathbf{E}_2\mathbf{E}_1$ . (There are other possible answers.)
12. $\mathbf{A}=\begin{bmatrix}-2 & 1\\4 & -1\end{bmatrix}$, span of the columns is $\mathbf{R}^2$ and the columns are linearly independent.
13. $\mathbf{A}=\begin{bmatrix}-2 & 1\\-2 & 1\end{bmatrix}$, the columns are linearly dependent, and the span of the columns is the set of all scalar multiples of $\begin{bmatrix}1\\1\end{bmatrix}$.

Please let me know if you find any errors in these answers. Thank you!

**Practice Exam:**

1. (20pt) Give an example of each of the following or explain in one or two sentences why it would be impossible to do so.

   (a) A homogeneous linear system with no solutions.
   (b) A linear system of four equations in three unknowns whose solution involves a free variable.
   (c) A linear system of three equations in four unknowns with a unique solution.
   (d) A linear system of three equations in three unknowns that has no solution.
   (e) An invertible matrix for which the corresponding homogeneous system has only the trivial solution.

2. (10pt) Consider the set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

   (a) Are these vectors linearly independent?
   (b) Do these vectors span $\mathbf{R}^3$?

3. (20pt) Consider the matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$.

   (a) Find elementary matrices $\mathbf{E}_1$ $\mathbf{E}_2$, and $\mathbf{E}_3$ such that $\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A}=\mathrm{I}$.
   (b) Write $\mathbf{A}^{-1}$as a product of elementary matrices.
   (c) Write $\mathbf{A}$ as a product of elementary matrices.

4. (20pt) A square matrix $\mathbf{A}$ is called symmetric if $\mathbf{A}^t=\mathbf{A}$ ($\mathbf{A}$ equals its transpose) and skew-symmetric if $\mathbf{A}^t=-\mathbf{A}$. Prove that if $\mathbf{B}$ is a square matrix, then $\mathbf{BB}^t$ and $\mathbf{B}+\mathbf{B}^t$ are each symmetric, while $\mathbf{B}-\mathbf{B}^t$ is skew-symmetric.

5. (10pt) Give an example of a 2x2 matrix $\mathbf{A}$ that is not the zero matrix, such that $\mathbf{A}^2$ equals the zero matrix.

6. (20pt) Consider the following system of linear equations: $\begin{aligned} x_1 + 3x_2 + 5x_3 &= b_1 \\ x_1 + 3x_2 + 4x_3 &= b_2 \\ x_1 + 3x_2 + 2x_3 &= b_3 \end{aligned}$.

   (a) What condition(s) must $b_1$, $b_2$, and $b_3$ satisfy for this system to have at least one solution?
   (b) Assuming the condition in part b holds, write down the general solution of this system as a linear combination of vectors.
   (c) State the solution set for the homogeneous system (with $b_1= b_2= b_3=0$).
   (d) Find a basis for the solution set to the homogeneous system.