A Sample Proof

Math 272, Spring 2012

Here is a sample problem, followed by two proofs.

Let U and V be subspaces of a vector space W. Their ***sum*** is defined to be

$$U + V = \{u + v \mid u \in U,\ v \in V\}.$$

Prove that $U + V$ is a subspace of W.

Here is the first proof.

Proof.
(1) Since U and V are subspaces of W, they both contain the zero vector $\mathbf{0}$ of W. Thus $\mathbf{0} \in U$ and $\mathbf{0} \in V$. Then, using the defining property of the zero vector, we obtain

$$\mathbf{0} = \mathbf{0} + \mathbf{0} \in U + V,$$

where the final $\in$ follows from the definition of $U + V$.

(2) Take $u, v \in U + V$. To show: $u + v \in U + V$.

First note that $u \in U + V$ implies that $u = u_1 + v_1$ for some $u_1 \in U$ and $v_1 \in V$ by the definition of $U + V$. Similarly, $v \in U + V$ implies that $v = u_2 + v_2$ for some $u_2 \in U$ and $v_2 \in V$. Then, using the commutative and associative properties of vector addition, we obtain

$$u + v = (u_1 + v_1) + (u_2 + v_2) = (u_1 + u_2) + (v_1 + v_2).$$

Since $u_1 + u_2 \in U$ (U is a subspace) and $v_1 + v_2 \in V$ (V is a subspace), we conclude that $u + v \in U + V$ by the definition of $U + V$.

(3) Take $u \in U + V$ and $c \in \mathbb{R}$. To show: $cu \in U + V$.

As in (2), $u \in U + V$ implies that $u = u_1 + v_1$ for some $u_1 \in U$ and $v_1 \in V$. Then, using one of the distributive properties of scalar multiplication, we obtain

$$cu = c(u_1 + v_1) = cu_1 + cv_1.$$

Since $cu_1 \in U$ (U is a subspace) and $cv_1 \in V$ (V is a subspace), we conclude that $cu \in U + V$ by the definition of $U + V$. QED

The second proof is the first plus comments in a [different font].

Proof.
(1) [To prove that $U+V$ contains the zero vector, you need to write zero as something in U plus something in V. You need to realize that the obvious way to do this is to use zero plus zero.] Since U and V are subspaces of W, they both contain the zero vector $\mathbf{0}$ of W. Thus $\mathbf{0} \in U$ and $\mathbf{0} \in V$. [Say explicitly that U, V contain the zero vector because they are subspaces.] Then, using the defining property of the zero vector [say explicitly why zero = zero + zero], we obtain

$$\mathbf{0} = \mathbf{0} + \mathbf{0} \in U + V,$$

where the final $\in$ follows from the definition of $U+V$. [Say explicitly that the above equation implies that zero is in $U+V$.]

(2) Take $u, v \in U+V$. To show: $u+v \in U+V$. [This is what it means for $U+V$ to be closed under addition.]

First note that $u \in U+V$ implies that $u = u_1 + v_1$ for some $u_1 \in U$ and $v_1 \in V$ by the definition of $U+V$. [This is a critical step—once you have $u \in U+V$, you need to immediately write down what this means. The definition of $U+V$ given on the first page writes elements of $U+V$ as $u+v$. But you can't use the same letters here since u and v are already taken. This is where u_1 and v_1 come from. Here is the key thing:

- Rather than just **repeating** the definition of $U+V$, you instead **act** on the definition as it applies to the particular element $u \in U+V$.]

Similarly, $v \in U+V$ implies that $v = u_2 + v_2$ for some $u_2 \in U$ and $v_2 \in V$. [Be sure you understand where u_2 and v_2 come from.] Then, using the commutative and associative properties of vector addition [in a proof, always cite the properties you are using], we obtain

$$u + v = (u_1 + v_1) + (u_2 + v_2) = (u_1 + u_2) + (v_1 + v_2).$$

[This is the key strategy of the proof: since you want to show $u+v \in U+V$, you start with $u+v$ and see where it leads.] Since $u_1 + u_2 \in U$ (U is a subspace) [say explicitly that U contains u_1+u_2 because it is a subspace.] and $v_1 + v_2 \in V$ (V is a subspace) [same], we conclude that $u+v \in U+V$ by the definition of $U+V$. [Say explicitly that the above equation implies that $u+v$ is in $U+V$.]

(3) Now look at the proof of (3) on page 1 and figure out what the comments are. Memorizing this proof is useless; rather, you need to absorb the strategy so that you can generate the proof on your own.