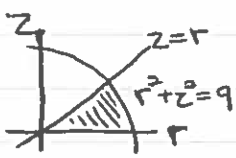
Solutions – Final Exam Math 211 Fall 2012

1. Find the plane containing 3 of the points and check that the 4^{th} point also lies on that plane. (There are also more direct ways to verify the points are coplanar.)
Normal vector to plane containing first 3 points: $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 3 & -1 \\ 1 & -2 & 0 \end{vmatrix} = \langle -2,-1,-13 \rangle$
Plane is $2x+y+13z=29$
Plug in $(-5,0,3)$: $2(-5)+0+13(3)=29$, so plane eq is satisfied.

2. (a) Velocity $\vec{r}\,'(t) = 3\sqrt{2}\,\hat{i} - 3e^{-3t}\hat{j} + 3e^{3t}\hat{k}$
Acceleration $\vec{r}\,''(t) = 0\hat{i} + 9e^{-3t}\hat{j} + 9e^{3t}\hat{k}$
(b) Speed $\|\vec{r}\,'(t)\| = \sqrt{3^2\cdot 2 + 3^2(e^{-3t})^2 + 3^2(e^{3t})^2} = 3\sqrt{(e^{-3t}+e^{3t})^2} = 3(e^{-3t}+e^{3t})$
(c) Distance traveled $= \int_0^1 \|\vec{r}\,'(t)\|dt = 3\left(-\frac{1}{3}e^{3t}+\frac{1}{3}e^{3t}\right) = e^3 - e^{-3}$

3. 

$$V = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^3 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 9\sqrt{2}\pi$$
$$V = \int_0^{2\pi}\int_0^{3/\sqrt{2}}\int_z^{\sqrt{9-z^2}} r \, dr \, dz \, d\theta = 9\sqrt{2}\pi$$

4. Let $u=x+y$ and $v=x-y$, so $x=\frac{1}{2}(u+v)$, $y=\frac{1}{2}(u-v)$
Jacobian is $\left|\begin{smallmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{smallmatrix}\right| = |-1/4-1/4| = 1/2$
Integral becomes $\frac{1}{2}\int_{-1}^1\int_1^4 u^2 e^v \, du \, dv = \frac{1}{2}\cdot\frac{1}{3}u^3\Big|_1^4 \cdot e^v\Big|_{-1}^1 = \frac{21}{2}\left(e-\frac{1}{e}\right)$

5. Find the critical points: $f_x = 2x-2xy=0$ (so $x=0$ or $y=1$)
$f_y = -x^2+4y=0$ (so $y=\frac{1}{4}x^2$)
$D = f_{xx}f_{yy} - f_{xy}^2 = (2-2y)(-2x)-16$
At $(0,0)$, $D=8>0$ and $f_{xx}=2>0$, so $(0,0)$ is a local minimum.
At $(2,1)$ and $(-2,1)$, $D=-16<0$, so $(\pm 2,1)$ are saddle points.

6. (a) $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{r\to 0^+} \frac{3r^2\cos^3\theta}{r} = \lim\limits_{r\to 0^+} 3r\cos^3\theta = 0 = f(0,0)$, so f is cont.
(b) $\frac{\partial f}{\partial x}(0,0) = \lim\limits_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0}\frac{3h^2/|h| - 0}{h} = \lim\limits_{h\to 0}\frac{3h}{|h|}$ DNE (3 as $h\to 0^+$ but -3 as $h\to 0^-$)
$\frac{\partial f}{\partial y}(0,0) = \lim\limits_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim\limits_{h\to 0}\frac{0/|h| - 0}{h} = 0$

7. $\nabla f = \langle 2x + \sin y,\ x\cos y \rangle$

$\nabla f(1,0) = \langle 2, 1 \rangle$

(a) $D_{\langle 3/5, 4/5 \rangle} f(1,0) = \langle 2,1 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{6}{5} + \frac{4}{5} = 2$

(b) Max value is $\|\nabla f\| = \sqrt{2^2+1^2} = \sqrt{5}$

(c) $f(x,y) \approx f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$

$= 1 + 2(x-1) + 1(y-0)$

$f(.99, .1) \approx 1 + 2(.99-1) + 1(.1-0) = 1.08$

8. Limit toward $(0,0)$ along the x-axis is 0, but along $y=x$ is $1/2$, so the limit does not exist.

9. The Extreme Value Theorem tells us that the absolute max & min will exist (the function is continuous and the region is closed and bounded), and will occur either at a critical point inside the region or on the boundary. The only critical point of $f(x,y)$ is $(0,0)$, which is not in the region (doesn't satisfy inequality). Use Lagrange Multipliers to determine extrema on boundary $(x-3)^2+y^2=5$: $\nabla f = \lambda \nabla g$ leads to $y = 2\lambda(x-3)$ and $x = 2\lambda y$.
Plug $\lambda = x/2y$ into 1st eq to obtain $y^2 = x(x-3)$.
Plug into contraint: $(x-3)^2 + x(x-3) = 5 \leftrightarrow 2x^2 - 9x + 4 = 0$

$\leftrightarrow (2x-1)(x-4) = 0$

$x = 1/2$ is never in region, so $x = 4$ and $y = \pm 2$.
$f(4,2) = 8$ is the abs max and $f(4,-2) = -8$ is the abs min.

10. $\oint_C F \cdot dr = \iint_{\text{interior of } C} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint 0\, dA = 0$

11. $\int_{\square} P\,dx + Q\,dy = 0 + 1 + 1 + 0 = 2$ (calculate along each side of square)

$\iint_{\blacksquare} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{\blacksquare} 2\, dA = 2(\text{area of square}) = 2$

12. Call point on circle (x_0, y_0), with $x_0^2 + y_0^2 = 1$, so $\vec{r}(t) = \langle (1-t)x_0, (1-t)y_0 - t \rangle$ for $0 \le t \le 1$

$\|\vec{r}\,'(t)\| = \sqrt{x_0^2 + (y_0+1)^2} = \sqrt{2+2y_0}$. Plugging into given expression,

$T = \int_0^1 \frac{\sqrt{2+2y_0}\, dt}{\sqrt{2g}\sqrt{y_0 - (1-t)y_0 + t}} = \int_0^1 \frac{\sqrt{2(1+y_0)}\, dt}{\sqrt{2g}\sqrt{t(y_0+1)}} = \int_0^1 \frac{dt}{\sqrt{gt}} = \frac{1}{\sqrt{g}} \cdot 2t^{1/2}\Big|_0^1 = \frac{2}{\sqrt{g}}$

is independent of y_0.