Fall 2014 Math 211 Exam 2 Review Sheet

The exam will be in class on Monday, November 3, and will cover Chapter 14. You will not be allowed use of a calculator or any other device other than your pencil or pen and some scratch paper. Notes are also not allowed. In kindness to your fellow test-takers, please turn off all cell phones and anything else that might beep or be a distraction.

Important topics:

- Level curves of $z=f(x,y)$
- Limits and continuity for functions of 2 variables (no ε-δ proofs on this exam)
- Computing partial derivatives from the definition (limit as h goes to 0 of the appropriate difference quotient)
- Partial derivatives
- Tangent plane to a surface
- Linear approximations to functions of 2 variables
- Chain rule
- Implicit differentiation of $F(x,y,z)=0$
- Computing directional derivatives and interpretation as steepness of surface
- Gradient and its importance (points in direction of greatest increase), and its relation to level sets (gradient perpendicular to level curve or level surface)
- Second Partials Test to find and classify critical points of a function (local max/min and saddle points)
- Optimization of multivariable functions (but not Lagrange multipliers)

Chapter 14 Review Exercises: 5, 13, 15, 19, 25, 27, 29, 31, 33, 37, 44, 45, 51, 53, 55

Answer to 44(d): $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \|\nabla f\| \|\mathbf{u}\| \cos\theta = \frac{1}{2}\|\nabla f\|$ and **u** is a unit vector, so the directional derivative is half of its maximum value (length of the gradient) when the angle between **u** and $\|\nabla f\|$ is $\theta=\cos^{-1}(1/2)=60$ degrees or $\pi/3$ radians.

Additional problems (answers on next page):

1. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2 - y^2}{x^2 + 2y^2} = ?$
2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(3x^2 + y^2)}{x^2 + 2y^2} = ?$
3. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^3 - y^3}{x^2 + y^2} = ?$
4. You are given only the following information about a differentiable function f:
   $f(3, 2) = 8, \quad f(3.01, 2) = 7.9, \quad f(3, 1.98) = 7.6.$
   a. Approximate the equation of the tangent plane to the surface $z=f(x,y)$ at (3, 2, 8).
   b. Use part (a) to estimate the value of $f(3.02, 1.96)$.

5. Let $f(x,y)=\begin{cases} \dfrac{5x^3+xy^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$
   a. Evaluate $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ using polar coordinates and state whether this function is continuous at (0,0).
   b. Find $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.
6. Find the directional derivative $\mathbf{D_u}f(2,3)$ for $f(x,y)=x^3y-3x^2$ in the same direction as the vector $\mathbf{v}=<3,-4>$. What can you say about the slope of the surface $z=f(x,y)$ at (2,3,12) in the direction given by **v**?
7. Let $f(x,y)=\begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$
   a. Evaluate $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ using polar coordinates and state whether this function is continuous at (0,0).
   b. Find $\dfrac{\partial f}{\partial x}(0,0)$ using the limit definition of partial derivative.
8. Find an equation for the tangent plane to the double-cone $z^2=x^2+y^2$ at the point (a,b,c), which could be any point on the cone (so your tangent plane equation will involve a, b, and c as well as x, y, and z). Use this equation to show that the tangent plane passes through the origin, no matter which point (a,b,c) we chose on the cone.

Answers: **1**. DNE (prove by finding two directions with different limits); **2**. DNE; **3**. 0 (can use polar coordinates); **4**. (a) z=8-10(x-3)+20(y-2) (b) 7.0; **5**. (a) Yes, f is continuous at (0,0) (b) Partials at (0,0) equal 5 and 0, respectively. **6**. $\mathbf{D_u}f(2,3)=8$, surface is sloping steeply up is this direction; **7**. (a) 0, is continuous, (b) 0. **8**. Note that $c^2=a^2+b^2$ since (a,b,c) is on the cone. The gradient of $F(x,y,z)=x^2+y^2-z^2$ is orthogonal to the surface $F(x,y,z)=0$, so a normal vector at (a,b,c) is $<2a,2b,-2c>$. The tangent plane is $ax+by-cz=0$, and the point (0,0,0) satisfies this equation so lies on the tangent plane.

Practice Exam

1. The equations $z=f(x,y)$ and $F(x,y,z)=f(x,y)-z=0$ describe the same surface in 3-space, yet ∇f and ∇F are not the same vectors.
   a. Write down ∇f and ∇F in terms of the partial derivatives of f.

   b. If $f(1,2)=5$ and $\nabla f(1,2)=2\mathbf{i}-3\mathbf{j}$, find a vector normal to the surface $z=f(x,y)$ at the point with x=1 and y=2, and write down an equation for the tangent plane to the surface at that point.

2. Find all points on the surface $z = 3x^2 - 4y^2$ where the vector <3,2,2> is perpendicular to the tangent plane.

3. Find the local maxima, local minima, and saddle points of the function $f(x, y) = x^2 + 6xy + 10y^2 - 4y + 4$.

4. Suppose $\nabla f = < -3,0 >$.
   a. In what direction(s) does the directional derivative $\mathbf{D_u}f$ take its maximum value?

   b. In what direction(s) does the directional derivative $\mathbf{D_u}f$ take its minimum value?

   c. In what direction(s) does the directional derivative $\mathbf{D_u}f$ equal 0?

   d. In what direction(s) does the directional derivative $\mathbf{D_u}f$ equal half of its minimum value?

5. Let $f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$
   a. Evaluate $\lim_{(x,y)\to(0,0)} f(x,y)$ using polar coordinates and state whether this function is continuous at (0,0).
   b. Find $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ using the limit definition of partial derivative.

Answers:

1. ∇F =<f_x,f_y,-1> but ∇f =<f_x,f_y>, $2x-3y-z+9=0$
2. (-1/4, 1/8, 1/8)
3. $f(-6,2)=0$ is the local minimum. There are no other local extrema or saddle points.
4. a. <-1,0)> b. <1,0> c. <0,1> and <0,-1> d. <1/2, $\pm\sqrt{3}/2$>
5. a. 0, continuous, b. $f_x(0,0)=1$ and $f_y(0,0)=-1$