Amherst College
Department of Mathematics

MATH 211 | FINAL EXAM | SPRING 2014

NAME: Solutions

Read This First!

- This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.
- You need *NOT* simplify algebraically complicated answers. However, numerical answers such as $\sin\frac{\pi}{6}$, $\arctan(\sqrt{3})$, $4^{3/2}$, $e^{\ln 4}$, $\ln e^7$, $e^{-\ln 5}$, $e^{3\ln 3}$, or $\cosh(\ln 3)$ should be simplified.
- Please read each question carefully. *Show all of your work* and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Grading - For Administrative Use Only

Question:	1	2	3	4	5	6	7	8	9	10	11	12	Total
Points:	6	8	8	8	10	8	8	10	8	12	6	8	100
Score:													

1. Find the angle between $u = \langle 2, 3, 1\rangle$ and $v = \langle 4, 1, 2\rangle$. [6]

$$\cos\theta = \frac{u \cdot v}{|u||v|}$$

$$= \frac{8+3+2}{\sqrt{4+9+1}\,\sqrt{16+1+4}}$$

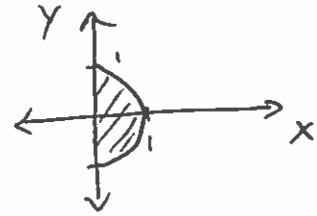
$$= \frac{13}{\sqrt{14}\,\sqrt{21}} = \frac{13\sqrt{6}}{42}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{13\sqrt{6}}{42}\right)}$$

2. Convert the following integral from rectangular to cylindrical coordinates. [8]
DO NOT INTEGRATE.

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-x^2-y^2}^{x^2+y^2} 21xy^2 dz\, dy\, dx$$

$$\left.\begin{array}{l} 0 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \end{array}\right\}$$



$$-\underbrace{(x^2+y^2)}_{r^2} \le z \le \underbrace{x^2+y^2}_{r^2}$$

$$0 \le r \le 1$$

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \int_{-r^2}^{r^2} 21 r^4 \cos\theta \sin^2\theta \, dz\, dr\, d\theta}$$

$$21xy^2 = 21 r\cos\theta\, r^2 \sin^2\theta$$

$$dz\,dy\,dx = r\, dz\, dr\, d\theta$$

3. Find the volume of the parallelepiped determined by $u = \langle 2, 2, -4\rangle$, $v = \langle -2, 0, -2\rangle$, and $w = \langle 4, 3, -4\rangle$. [8]

$$V = |u \cdot (v \times w)| = \left| \begin{vmatrix} 2 & 2 & -4 \\ -2 & 0 & -2 \\ 4 & 3 & -4 \end{vmatrix} \right|$$

$$= |2(6) - 2(8+8) - 4(-6)|$$

$$= |12 - 32 + 24|$$

$$= \boxed{4}$$

4. Find the equation of the plane tangent to the surface [8]

$$z = \ln(2x + y)$$

at the point $(-1, 3)$.

$z = f(x, y) = \ln(2x+y)$

$f_x = \dfrac{2}{2x+y}$ $\qquad$ $f_x(-1, 3) = 2$

$f_y = \dfrac{1}{2x+y}$ $\qquad$ $f_y(-1, 3) = 1$

$z = 2(x+1) + 1(y-3)$

$$\boxed{z = 2x + y - 1}$$

5. Find the volume of the region cut from the solid sphere $\rho \leq 1$ by the half planes $\theta = 0$ and $\theta = \dfrac{\pi}{6}$ in the first octant. [10]

$$0 \leq \rho \leq 1$$

$$0 \leq \theta \leq \frac{\pi}{6}$$

$$0 \leq \phi \leq \frac{\pi}{2}$$

$$V = \int_0^{\frac{\pi}{6}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \int_0^{\frac{\pi}{2}} \left(\frac{1}{3}\right) \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{6}} d\theta$$

$$= \boxed{\frac{\pi}{18}}$$

6. A function is called 'Harmonic' if $f_{xx} + f_{yy} + f_{zz} = 0$. Show that the function [8]

$$f(x, y, z) = 7e^{x+2y} \sin\left(z\sqrt{5}\right)$$

is Harmonic.

$$f_x = 7e^{x+2y} \sin(z\sqrt{5})$$

$$f_{xx} = 7e^{x+2y} \sin(z\sqrt{5})$$

$$f_y = 14e^{x+2y} \sin(z\sqrt{5})$$

$$f_{yy} = 28e^{x+2y} \sin(z\sqrt{5})$$

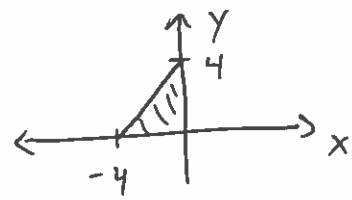
$$f_z = 7\sqrt{5}\, e^{x+2y} \cos(z\sqrt{5})$$

$$f_{zz} = -35e^{x+2y} \sin(z\sqrt{5})$$

$$\boxed{\begin{aligned} f_{xx} + f_{yy} + f_{zz} &= (7 + 28 - 35)e^{x+2y} \sin(z\sqrt{5}) \\ &= 0 \ \checkmark \end{aligned}}$$

7. Find the centroid of the triangular region cut from the **second quadrant** by the line $y - x = 4$. [8]

$\rho = 1$



$$m = \int_{-4}^{0}\int_{0}^{x+4} dy\,dx$$

$$= \int_{-4}^{0} (x+4)\,dx = 8$$

$$M_x = \int_{-4}^{0}\int_{0}^{x+4} y\,dy\,dx$$

$$= \int_{-4}^{0} \frac{1}{2}\left(x^2+8x+16\right) dx = \frac{32}{3}$$

$$M_y = \int_{-4}^{0}\int_{0}^{x+4} x\,dy\,dx$$

$$= \int_{-4}^{0} (x^2+4x)\,dx = -\frac{32}{3}$$

Centroid: $\left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \boxed{\left(-\frac{4}{3}, \frac{4}{3}\right)}$

8. For each of the following, find the limit or show that the limit does not exist.

(a) $\displaystyle\lim_{(x,y)\to(4,0)} \frac{xy-4y}{(x-4)^2+y^2}$ [5]

Path 1 : $y = x-4$

$$\lim_{x\to 4} \frac{(x-4)^2}{2(x-4)^2} = \frac{1}{2}$$

Path 2 : $y = 2(x-4)$

$$\lim_{x\to 4} \frac{2(x-4)^2}{5(x-4)^2} = \frac{2}{5}$$

$\frac{1}{2} \neq \frac{2}{5}$ Thus by the 2 Path Test, the limit does NOT exist.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-3y^3}{\sqrt{x^2+y^2}}$ [5]

$0 \le \left| \frac{x^2}{\sqrt{x^2+y^2}} \right| \le |x|$ Thus by the Squeeze Thm, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0$

$0 \le \left| \frac{-3y^3}{\sqrt{x^2+y^2}} \right| \le 3|y^2|$ Thus by the Squeeze Thm, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{-3y^3}{\sqrt{x^2+y^2}} = 0$

Therefore

$$\lim_{(x,y)\to(0,0)} \frac{x^2-3y^3}{\sqrt{x^2+y^2}} = 0$$

9. Find the work done by a force field $F = xyi + yzj + xzk$ from $(0,0,0)$ to $(1,1,1)$ over the path given by $r(t) = ti + t^2 j + t^4 k$. [8]

$$\begin{cases} x(t) = t \\ y(t) = t^2 \\ z(t) = t^4 \end{cases}$$

$$F(r(t)) = \langle t^3, t^6, t^5 \rangle$$

$$r'(t) = \langle 1, 2t, 4t^3 \rangle$$

$$F(r(t)) \cdot r'(t) = t^3 + 2t^7 + 4t^8$$

$\Rightarrow$ From $(0,0,0)$ to $(1,1,1)$ $\Rightarrow$ $0 \leq t \leq 1$

$$W = \int_0^1 (t^3 + 2t^7 + 4t^8)\, dt$$

$$= \frac{t^4}{4} + \frac{t^8}{4} + \frac{4t^9}{9} \Big]_0^1$$

$$= \boxed{\frac{17}{18}}$$

10. Consider the vector field $F = (2xy^4 - \cos y)\, i + (4x^2y^3 + 1 + x\sin y)\, j$.

(a) Show that the vector field is conservative. [4]

$$\frac{\partial P}{\partial y} = 8xy^3 + \sin y$$

$$\frac{\partial Q}{\partial x} = 8xy^3 + \sin y$$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\boxed{F \text{ is conservative}}$

(b) Find a potential function corresponding to F. [4]

$$f_x = 2xy^4 - \cos y \implies f(x,y) = x^2y^4 - x\cos y + g(y)$$

$$f_y = 4x^2y^3 + 1 + x\sin y \implies f(x,y) = x^2y^4 + y - x\cos y + h(x)$$

Thus

$$\boxed{f(x,y) = x^2y^4 + y - x\cos y}$$

(c) Evaluate the integral [4]

$$\int_C (2xy^4 - \cos y)\, dx + (4x^2y^3 + 1 + x\sin y)\, dy$$

where C is a smooth curve from $(3,1)$ to $\left(2, \frac{\pi}{2}\right)$.

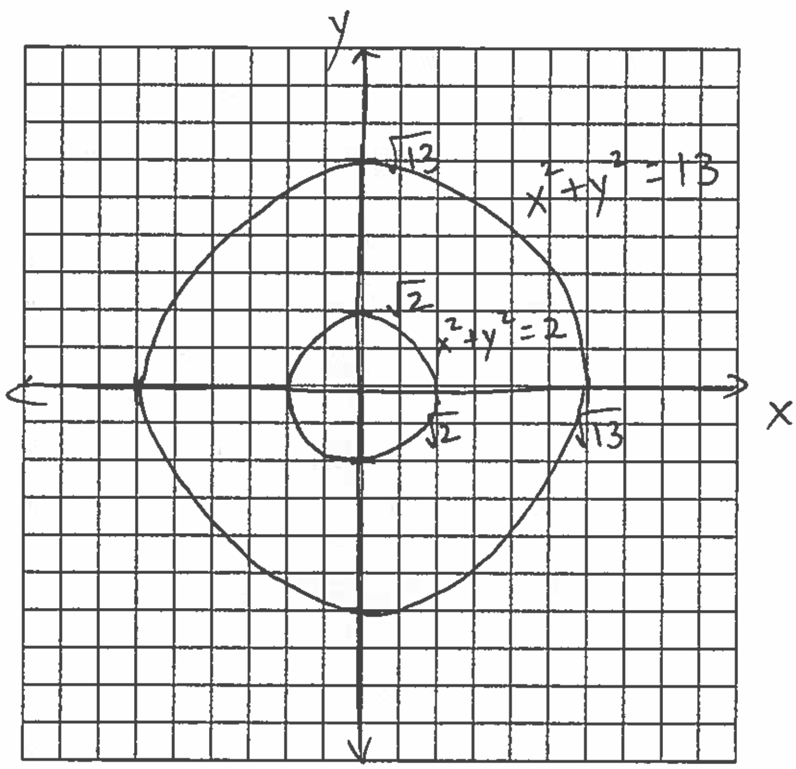
$$f\left(2, \tfrac{\pi}{2}\right) - f(3,1)$$

$$= \boxed{\frac{\pi^4}{4} + \frac{\pi}{2} - 10 + 3\cos(1)}$$

11. Given $f(x,y) = \sqrt{29 - x^2 - y^2}$, sketch the level curves that pass through the points $(2, -3, 4)$ and $(1, 1, 3\sqrt{3})$. Make sure to label your axes and tick marks. [6]

$$4 = \sqrt{29 - x^2 - y^2}$$
$$16 = 29 - x^2 - y^2$$
$$x^2 + y^2 = 13$$

$$3\sqrt{3} = \sqrt{29 - x^2 - y^2}$$
$$27 = 29 - x^2 - y^2$$
$$x^2 + y^2 = 2$$



12. Consider the function $f(x, y) = x^2 + 4y^2$.

(a) Find the directional derivative of f at the point $(3, 1)$ in the direction of the vector $\langle 1, -1\rangle$. [6]

$$\nabla f = \langle 2x, 8y \rangle$$

$$\nabla f(3,1) = \langle 6, 8 \rangle$$

$$u = \frac{\langle 1, -1 \rangle}{|\langle 1, -1 \rangle|} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

$$D_u f = \nabla f \cdot u = \frac{6}{\sqrt{2}} - \frac{8}{\sqrt{2}} = \boxed{-\sqrt{2}}$$

(b) In what direction is the directional derivative greatest at $(3, 1)$? [2]

$$\langle 6, 8 \rangle$$