

Fall 2012 Math 211 Exam 2 Review Sheet

The exam will be in class on Monday, November 5, and will cover Chapter 14. You will not be allowed use of a calculator or any other device other than your pencil or pen and some scratch paper. Notes are also not allowed. In kindness to your fellow test-takers, please turn off all cell phones and anything else that might beep or be a distraction.

Important topics:

- Level curves of $z=f(x,y)$
- Limits and continuity for functions of 2 variables (no ϵ - δ proofs)
- Computing partial derivatives from the definition (limit as h goes to 0 of the appropriate difference quotient)
- Partial derivatives
- Tangent plane to a surface
- Linear approximations to functions of 2 variables
- Chain rule
- Implicit differentiation of $F(x,y,z)=0$
- Computing directional derivatives and interpretation as steepness of surface
- Gradient and its importance (points in direction of greatest increase), and its relation to level sets (gradient perpendicular to level curve or level surface)
- Second Partials Test to find and classify critical points of a function (local max/min and saddle points)
- Optimization of multivariable functions
- Method of Lagrange multipliers for constrained optimization

Chapter 14 Review Exercises: 5, 13, 15, 19, 25, 27, 29, 31, 33, 37, 44, 45, 51, 53, 55, 59, 61, 63

Answer to 44(d): $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \|\nabla f\| \|\mathbf{u}\| \cos \theta = \frac{1}{2} \|\nabla f\|$ and \mathbf{u} is a unit vector, so the directional derivative is half of its maximum value (length of the gradient) when the angle between \mathbf{u} and $\|\nabla f\|$ is $\theta = \cos^{-1}(1/2) = 60$ degrees or $\pi/3$ radians.

Additional problems (answers on next page):

1. $\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 - y^2}{x^2 + 2y^2} = ?$

2. $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(3x^2 + y^2)}{x^2 + 2y^2} = ?$

3. $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^3 - y^3}{x^2 + y^2} = ?$

4. You are given only the following information about a differentiable function f :

$$f(3, 2) = 8, \quad f(3.01, 2) = 7.9, \quad f(3, 1.98) = 7.6.$$

- Approximate the equation of the tangent plane to the surface $z=f(x,y)$ at $(3, 2, 8)$.
- Use part (a) to estimate the value of $f(3.02, 1.96)$.

5. Let $f(x,y) = \begin{cases} \frac{5x^3 + xy^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$
- Evaluate $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ using polar coordinates and state whether this function is continuous at $(0,0)$.
 - Find $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.
6. Find the directional derivative $\mathbf{D}_{\mathbf{v}}f(2,3)$ for $f(x,y) = x^3y - 3x^2$ in the same direction as the vector $\mathbf{v} = \langle 3, -4 \rangle$. What can you say about the slope of the surface $z=f(x,y)$ at $(2,3,12)$ in the direction given by \mathbf{v} ?
7. Let $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$
- Evaluate $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ using polar coordinates and state whether this function is continuous at $(0,0)$.
 - Find $\frac{\partial f}{\partial x}(0,0)$ using the limit definition of partial derivative.
8. Find an equation for the tangent plane to the double-cone $z^2 = x^2 + y^2$ at the point (a,b,c) , which could be any point on the cone (so your tangent plane equation will involve a, b , and c as well as x, y , and z). Use this equation to show that the tangent plane passes through the origin, no matter which point (a,b,c) we chose on the cone.

Answers: **1.** DNE (prove by finding two directions with different limits); **2.** DNE; **3.** 0 (can use polar coordinates); **4.** (a) $z=8-10(x-3)+20(y-2)$ (b) 7.0; **5.** (a) Yes, f is continuous at $(0,0)$ (b) Partial derivatives at $(0,0)$ equal 5 and 0, respectively. **6.** $\mathbf{D}_{\mathbf{v}}f(2,3)=8$, surface is sloping steeply up in this direction; **7.** (a) 0, is continuous, (b) 0. **8.** Note that $c^2=a^2+b^2$ since (a,b,c) is on the cone. The gradient of $F(x,y,z)=x^2+y^2-z^2$ is orthogonal to the surface $F(x,y,z)=0$, so a normal vector at (a,b,c) is $\langle 2a, 2b, -2c \rangle$. The tangent plane is $ax+by-cz=0$, and the point $(0,0,0)$ satisfies this equation so lies on the tangent plane.

Practice Exam

- The equations $z = f(x,y)$ and $F(x,y,z) = f(x,y) - z = 0$ describe the same surface in 3-space, yet ∇f and ∇F are not the same vectors.
 - Write down ∇f and ∇F in terms of the partial derivatives of f .
 - If $f(1,2) = 5$ and $\nabla f(1,2) = 2\mathbf{i} - 3\mathbf{j}$, find a vector normal to the surface $z = f(x,y)$ at the point with $x=1$ and $y=2$, and write down an equation for the tangent plane to the surface at that point.

2. Find all points on the surface $z = 3x^2 - 4y^2$ where the vector $\langle 3, 2, 2 \rangle$ is perpendicular to the tangent plane.
3. Find the local maxima, local minima, and saddle points of the function $f(x, y) = x^2 + 6xy + 10y^2 - 4x + 4$.
4. Suppose $\nabla f = \langle -3, 0 \rangle$.
 - a. In what direction(s) does the directional derivative $D_{\mathbf{u}}f$ take its maximum value?
 - b. In what direction(s) does the directional derivative $D_{\mathbf{u}}f$ take its minimum value?
 - c. In what direction(s) does the directional derivative $D_{\mathbf{u}}f$ equal 0?
 - d. In what direction(s) does the directional derivative $D_{\mathbf{u}}f$ equal half of its minimum value?
5. Let $f(x, y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$
 - a. Evaluate $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ using polar coordinates and state whether this function is continuous at $(0, 0)$.
 - b. Find $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ using the limit definition of partial derivative.

Answers:

1. $\nabla F = \langle f_x, f_y, -1 \rangle$ but $\nabla f = \langle f_x, f_y \rangle$, $2x - 3y - z + 9 = 0$
2. $(-1/4, 1/8, 1/8)$
3. $f(-6, 2) = 0$ is the local minimum. There are no other local extrema or saddle points.
4. a. $\langle -1, 0 \rangle$ b. $\langle 1, 0 \rangle$ c. $\langle 0, 1 \rangle$ and $\langle 0, -1 \rangle$ d. $\langle 1/2, \pm\sqrt{3}/2 \rangle$
5. a. 0, continuous, b. $f_x(0, 0) = 1$ and $f_y(0, 0) = -1$