Non-archimedean Dynamics in Dimension One: Lecture 1

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Non-archimedean Fields

Let $K$ be a field with a non-archimedean absolute value $| \cdot | : K \to \mathbb{R}$.

That is, for all $x, y \in K$,

- $|x| \geq 0$, with equality iff $x = 0$,
- $|xy| = |x| \cdot |y|$,
- $|x + y| \leq \max\{ |x|, |y| \}$.

We assume $| \cdot |$ is nontrivial; that is, $|K| \not\supseteq \{0, 1\}$.

We usually assume $K$ is complete w.r.t. $| \cdot |$.
(All Cauchy sequences converge).

**Fun Fact:** Let $K$ be a complete non-archimedean field, and let $\{a_n\}_{n \geq 0}$ be a sequence in $K$. Then

$$\sum_{n \geq 0} a_n \text{ converges if and only if } \lim_{n \to \infty} a_n = 0.$$
The Residue Field and Value Group

Let $K$ be a non-archimedean field. The ring of integers and (unique) maximal ideal of $K$ are

$$\mathcal{O}_K = \{x \in K : |x| \leq 1\} \quad \text{and} \quad \mathcal{M}_K = \{x \in K : |x| < 1\}.$$

The residue field of $K$ is

$$k := \mathcal{O}_K / \mathcal{M}_K.$$

The value group of $K$ is

$$|K^\times| \subseteq (0, \infty).$$
A Sketch of a Non-archimedean Field with $k \cong \mathbb{F}_3$
Let $K$ be a complete non-archimedean field, and let $L/K$ be an algebraic extension.

Then $\ | \cdot \ |$ extends uniquely to $L$.

The new residue field $\ell$ is an algebraic extension of $k$.

The new value group $|L^\times|$ contains $|K^\times|$ as a subgroup.

The algebraic closure $\overline{K}$ of $K$ may not be complete.

But its completion $\mathbb{C}_K$ is both complete and algebraically closed.
Example: \( p \)-adic numbers

Fix \( p \geq 2 \) prime. The \( p \)-adic absolute value on \( \mathbb{Q} \) is given by

\[
\left| \frac{r}{s} p^n \right|_p = p^{-n} \quad \text{for } r, s \in \mathbb{Z} \text{ not divisible by } p.
\]

Idea: numbers divisible by large powers of \( p \) are “small”.

\[
\mathbb{Q}_p := \left\{ \sum_{n \geq n_0} a_n p^n : n_0 \in \mathbb{Z}, a_n \in \{0, 1, \ldots, p - 1\} \right\}
\]

is the completion of \( \mathbb{Q} \) w.r.t. \(| \cdot |_p\), with ring of integers

\[
\mathbb{Z}_p := \mathcal{O}_{\mathbb{Q}_p} = \left\{ \sum_{n \geq 0} a_n p^n : a_n \in \{0, 1, \ldots, p - 1\} \right\},
\]

maximal ideal \( \mathcal{M}_{\mathbb{Q}_p} := p\mathbb{Z}_p \), value group \( |\mathbb{Q}_p^\times|_p = p^{\mathbb{Z}} \), and residue field \( \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p \).

The completion \( \mathbb{C}_p \) of an algebraic closure \( \overline{\mathbb{Q}}_p \) has residue field \( \overline{\mathbb{F}}_p \) and value group \( |\mathbb{C}_p^\times| = p^{\mathbb{Q}} \).
Example: Laurent and Puiseux Series

Fix $\mathbb{F}$ a field. The field of formal Laurent series

$$\mathbb{F}((t)) := \left\{ \sum_{n \geq n_0} a_n t^n : n_0 \in \mathbb{Z}, a_n \in \mathbb{F} \right\}$$

has a non-archimedean absolute value

$$|f| := \varepsilon^{\text{ord}_{t=0} f},$$

where $0 < \varepsilon < 1$ is any (fixed) thing you want.

The ring of integers is the ring $\mathbb{F}[[t]]$ of power series, with maximal ideal $t\mathbb{F}[[t]]$, residue field

$$k = \mathbb{F}[[t]]/t\mathbb{F}[[t]] \cong \mathbb{F},$$

and value group $|\mathbb{F}((t))^{\times}| = \varepsilon^\mathbb{Z}$.

The completion $\mathbb{L}$ of an algebraic closure $\overline{\mathbb{F}((t))}$ is the field of formal Puiseux series over $\mathbb{F}$, with residue field $\overline{\mathbb{F}}$ and value group $|\mathbb{L}^{\times}| = \varepsilon^\mathbb{Q}$. 
Disks

Given \( a \in C_\mathcal{K} \) and \( r > 0 \),

\[
D(a, r) := \{ x \in C_\mathcal{K} : |x - a| < r \}
\]

and

\[
\overline{D}(a, r) := \{ x \in C_\mathcal{K} : |x - a| \leq r \}
\]

are the associated open disk and closed disk.

- if \( r \notin |C_\mathcal{K}_x| \), then \( D(a, r) = \overline{D}(a, r) \) is an irrational disk
- if \( r \in |C_\mathcal{K}_x| \), then then \( D(a, r) \subset \overline{D}(a, r) \).
- \( D(a, r) \) is a rational open disk
- \( \overline{D}(a, r) \) is a rational closed disk

Note:

- All disks are (topologically) both open and closed
- Any disk is exactly one of: rational open, rational closed, or irrational (as a disk).
More about Disks

- Any point of a disk is a center:
  \[ D(a, r) = D(b, r) \] (resp., \( \overline{D}(a, r) = \overline{D}(b, r) \))
  for all \( b \in D(a, r) \) (resp., \( b \in \overline{D}(a, r) \))

- Since our disks lie in \( \mathbb{C}_K \), and \( |\mathbb{C}_K^\times| \) is dense in \((0, \infty)\),
  the **radius** of a disk \( D \subseteq \mathbb{C}_K \) is well-defined,
  and equal to the diameter \( \sup \{ |x - y| : x, y \in D \} \).

- Two disks intersect if and only if one contains the other.

- All non-archimedean fields are totally disconnected.
  (I.e., the only connected nonempty subsets are singletons.)

- \( \mathbb{Q}_p \) and \( \mathbb{F}_q((t)) \) are locally compact,
  but \( \mathbb{C}_K \) is not locally compact.
Theorem

Let $a \in \mathbb{C}_K$ and $r > 0$.

Let $g(z) = c_0 + c_1(z - a) + \cdots + c_M(z - a)^M \in \mathbb{C}_K[z]$ be a polynomial. (Or more generally, $g(z) \in \mathbb{C}_K[[z - a]]$ is a power series satisfying certain mild convergence conditions)

Let $s := \max\{ |c_n| r^n \}$, and

\[ i := \text{minimum } n \geq 1 \text{ for which } |c_n| r^n = s, \]
\[ j := \text{maximum } n \geq 1 \text{ for which } |c_n| r^n = s. \]

Then $g$ maps

$D(a, r)$ $i$-to-1 onto $D(c_0, s)$, and

$D(a, r)$ $j$-to-1 onto $D(c_0, s)$,

counting multiplicity.
Example

\( \mathbb{C}_K = \mathbb{C}_p \), and \( g(z) = p^4z^5 + p^2z^3 + z^2 + pz + p^3 \).

Then for any \( r > 0 \), \( g(D(0, r)) = D(p^3, s) \), where

\[
s = \begin{cases}  
|p|_p r = p^{-1}r & \text{if } 0 < r \leq |p|_p = \frac{1}{p}, \\
r^2 & \text{if } \frac{1}{p} = |p|_p < r \leq |p|^{-4/3}_p = p^{4/3}, \\
|p^4|_p r^5 = p^{-4}r^5 & \text{if } r \geq |p|^{-4/3}_p = p^{4/3}.
\end{cases}
\]

[Note: \( D(p^3, s) = D(0, s) \) for \( s \geq |p|_p^3 = p^{-3} \).]

The mapping is 1-1 for \( r < |p|_p \),

2-1 for \( |p|_p \leq r < |p|^{-4/3}_p \),

5-1 for \( r \geq |p|^{-4/3}_p \).
**$\mathbb{P}^1(\mathbb{C}_K)$-Disks**

Recall $\mathbb{P}^1(\mathbb{C}_K) = \mathbb{C}_K \cup \{\infty\}$.

**Definition**

A $\mathbb{P}^1(\mathbb{C}_K)$-disk is either

- a disk $D \subseteq \mathbb{C}_K$, or
- the complement $\mathbb{P}^1(\mathbb{C}_K) \setminus D$ of a disk $D \subseteq \mathbb{C}_K$.

We can attach the adjectives *rational open*, *rational closed*, or *irrational* in the obvious way.

**Theorem**

Let $g(z) \in \mathbb{C}_K(z)$ be a non-constant rational function, and let $D \subseteq \mathbb{P}^1(\mathbb{C}_K)$ be a $\mathbb{P}^1(\mathbb{C}_K)$-disk. Then $g(D)$ is either

- all of $\mathbb{P}^1(\mathbb{C}_K)$, or
- a $\mathbb{P}^1(\mathbb{C}_K)$-disk of the same type as $D$. 
**Connected Affinoids**

**Definition**
A *connected affinoid* in $\mathbb{P}^1(\mathbb{C}_K)$ is a nonempty intersection of finitely many $\mathbb{P}^1(\mathbb{C}_K)$-disks. Equivalently, a connected affinoid is $\mathbb{P}^1(\mathbb{C}_K)$ with finitely many $\mathbb{P}^1(\mathbb{C}_K)$-disks removed.

We can attach the adjectives *rational open*, *rational closed*, or *irrational* in the obvious way.

**Theorem**
Let $g(z) \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 1$, and let $U \subseteq \mathbb{P}^1(\mathbb{C}_K)$ be a connected affinoid. Then

- $g(U)$ is either $\mathbb{P}^1(\mathbb{C}_K)$ or a connected affinoid of the same type as $U$.
- $g^{-1}(U)$ is a union of $1 \leq \ell \leq d$ connected affinoids $V_1, \ldots, V_{\ell}$ of the same type, and $g : V_i \rightarrow U$ is $d_i$-to-1, where $1 \leq d_i \leq d$, and $\sum_{i=1}^{\ell} d_i = d$. 
A Polynomial Example

\[ \mathbb{C}_K = \mathbb{C}_p, \text{ and } g(z) = pz^3 - z^2 + z. \] Then

- Let \( U \) be the rational closed annulus \( \overline{D}(0, 1) \setminus D(0, 1). \) Then \( g(U) = \overline{D}(0, 1). \)

  [Note: some points map 1-to-1, but others map 2-to-1.]

- \( g^{-1}(\overline{D}(0, 1)) = \overline{D}(0, 1) \cup \overline{D}(1/p, |p|_p), \) with
  - \( g : \overline{D}(0, 1) \to \overline{D}(0, 1) \) mapping 2-to-1, and
  - \( g : \overline{D}(1/p, |p|_p) \to \overline{D}(0, 1) \) mapping 1-to-1.

- \( g^{-1}(\overline{D}(0, |p|_p^{-3})) = \overline{D}(0, |p|_p^{-4/3}), \) mapping 3-to-1.
A Rational Example

\( \mathbb{C}_K \) is any complete, algebraically closed non-archimedean field, and
\( h(z) = z - \frac{1}{z} = \frac{z^2 - 1}{z} \).

\( h^{-1}(D(0, 1)) = D(1, 1) \cup D(-1, 1) \), with

\( \triangleright \) each of \( D(\pm 1, 1) \) mapping 1-1 onto \( D(0, 1) \) if the residue characteristic is not 2, or

\( \triangleright \) \( D(-1, 1) = D(1, 1) \) mapping 2-1 onto \( D(0, 1) \) if the residue characteristic is 2.

\( h^{-1}(\overline{D}(0, 1)) \) is the annulus \( \overline{D}(0, 1) \setminus D(0, 1) \), which maps 2-to-1 onto \( \overline{D}(0, 1) \).
Dynamics on $\mathbb{P}^1(\mathbb{C}_K)$: Classifying Periodic Points

Fix a rational function $\phi(z) \in \mathbb{C}_K(z)$ of degree $d \geq 2$.

If $x \in \mathbb{P}^1(\mathbb{C}_K)$ is periodic of exact period $n$, then $\lambda := (\phi^n)'(x)$ is the multiplier of $x$. We say $x$ is

- **attracting** if $|\lambda| < 1$.
- **repelling** if $|\lambda| > 1$.
- **indifferent** (or neutral) if $|\lambda| = 1$.

**Note:**

- The multiplier is the same for all points in the periodic cycle of $x$.
- The multiplier is coordinate-independent.
The Spherical Metric on $\mathbb{P}^1(\mathbb{C}_K)$

There is a spherical metric on $\mathbb{P}^1(\mathbb{C}_K)$ analogous to that on $\mathbb{P}^1(\mathbb{C})$:

$$\Delta(z_1, z_2) := \frac{|z_1 - z_2|}{\max\{1, |z_1|\} \max\{1, |z_2|\}}$$

More precisely, to allow the point at $\infty$, in homogeneous coordinates we write:

$$\Delta([x_1, y_1], [x_2, y_2]) := \frac{|x_1y_2 - x_2y_1|}{\max\{|x_1|, |y_1|\} \max\{|x_2|, |y_2|\}}$$
Fatou and Julia Sets

Definition
Let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$.
The (classical) Fatou set $\mathcal{F} = \mathcal{F}_\phi$ of $\phi$ is

$$\mathcal{F} = \{ x \in \mathbb{P}^1 : \{\phi^n\}_{n \geq 0} \text{ is equicontinuous on a neighborhood of } x \}$$

$$= \{ x \in \mathbb{P}^1 : \text{for all } n \geq 1 \text{ and } y \in \mathbb{P}^1(\mathbb{C}_K) \text{ s.t. } \Delta(x, y) \text{ is small, } \Delta(\phi^n(x), \phi^n(y)) \text{ is also small.} \}$$

The (classical) Julia set $\mathcal{J} = \mathcal{J}_\phi$ is $\phi$ is $\mathcal{J} = \mathbb{P}^1(\mathbb{C}_K) \setminus \mathcal{F}$.

Idea:
- In the Fatou set, small errors stay small under iteration.
- In the Julia set, small errors may become large.
Basic Properties of Fatou and Julia Sets

For both $\mathbb{C}$ and $\mathbb{C}_K$:

- $\mathcal{F}$ is open, and $\mathcal{J}$ is closed.
- $\mathcal{F}_{\phi^n} = \mathcal{F}_{\phi}$, and $\mathcal{J}_{\phi^n} = \mathcal{J}_{\phi}$.
- $\phi(\mathcal{F}) = \mathcal{F} = \phi^{-1}(\mathcal{F})$, and $\phi(\mathcal{J}) = \mathcal{J} = \phi^{-1}(\mathcal{J})$.
- All attracting periodic points are Fatou.
- All repelling periodic points are Julia.

An equivalent definition for $\mathbb{C}_K$:

Theorem

Let $\phi \in \mathbb{C}_K(z)$, and let $x \in \mathbb{P}^1(\mathbb{C}_K)$. Then $x \in \mathcal{F}_{\phi}$ if and only if there is a $\mathbb{P}^1(\mathbb{C}_K)$-disk $D \ni x$ such that

$$\#\mathbb{P}^1(\mathbb{C}_K) \setminus \left[ \bigcup_{n \geq 0} \phi^n(D) \right] \geq 2.$$
A Quadratic Example

\[ \phi(z) = z^2 + az \in \mathbb{C}_K[z]. \]

\[ \text{◮ If } |a| \leq 1, \text{ then } \phi(D(0, 1)) \subseteq D(0, 1), \]
and \( \phi(\mathbb{P}^1(\mathbb{C}_K) \setminus D(0, 1)) \subseteq \mathbb{P}^1(\mathbb{C}_K) \setminus D(0, 1). \)

So \( \mathcal{F}_\phi = \mathbb{P}^1(\mathbb{C}_K), \) and \( \mathcal{J}_\phi = \emptyset. \)

\[ \text{◮ If } |a| = R > 1, \text{ set } U_0 = D(0, R). \]
Then \( \phi(\mathbb{P}^1(\mathbb{C}_K) \setminus U_0) \subseteq \mathbb{P}^1(\mathbb{C}_K) \setminus U_0, \) so \( \mathbb{P}^1(\mathbb{C}_K) \setminus U_0 \subseteq \mathcal{F}_\phi. \)

For all \( n \geq 1, \) set \( U_n := \phi^{-n}(U_0). \)
Then \( U_n \) is a disjoint union of \( 2^n \) closed disks of radius \( R^{1-n}. \)

\[ \mathcal{J}_\phi = \bigcap_{n \geq 0} U_n \text{ is a Cantor set, and all points of } \]
\[ \mathcal{F}_\phi = \mathbb{P}^1(\mathbb{C}_K) \setminus \mathcal{J}_\phi \text{ are attracted to } \infty \text{ under iteration.} \]

Similarly: Over \( \mathbb{C}_p, \) Smart and Woodcock showed \( \phi(z) = (z^p - z)/p \) has \( \mathcal{J}_\phi = \mathbb{Z}_p. \)
A Cubic Example (due to Hsia)

Assume the residue characteristic is not 2, and set

\[ \phi(z) = az^3 + z^2 + bz + c, \quad \text{where } 0 < |a| < 1, \text{ and } |b|, |c| \leq 1. \]

Then \( \phi(\overline{D}(0, 1)) \subseteq \overline{D}(0, 1) \), so \( \overline{D}(0, 1) \subseteq \mathcal{F}_\phi \).

But \( \phi \) has a repelling fixed point \( \alpha \) with \( |\alpha| = |a|^{-1} > 1 \).

For all \( n \geq 1 \), there is a point \( \beta_n \in \phi^{-n}(\alpha) \) s.t. \( |\beta_n| = |a|^{-1/2^n} \).

Since \( \beta_n \in \mathcal{J}_\phi \), the set \( \mathcal{J}_\phi \) is not compact!!

Note: if we set \( U_0 = \overline{D}(0, |a|^{-1}) \), then

\[ \phi(\mathbb{P}^1(\mathbb{C}_K) \setminus U_0) \subseteq \mathbb{P}^1(\mathbb{C}_K) \setminus U_0 \]

as before, and \( U_n := \phi^{-n}(U_0) \) is a disjoint union of many disks.

In fact, \( \mathcal{F}_\phi \) is the union of \( \mathbb{P}^1(\mathbb{C}_K) \setminus \bigcap_{n \geq 1} U_n \) and all preimages of \( \overline{D}(0, 1) \).
## Contrasts with $\mathbb{C}$

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<tr>
<th>$\mathbb{C}$</th>
<th>$\mathbb{C}_K$</th>
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<tbody>
<tr>
<td>Some indifferent points are</td>
<td><strong>All</strong> indifferent points are Fatou</td>
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<td>Fatou, and some are Julia.</td>
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<tr>
<td>$\mathcal{J}$ is compact</td>
<td>$\mathcal{J}$ may not be compact</td>
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<td>$\mathcal{J}$ is nonempty</td>
<td>$\mathcal{J}$ may be empty</td>
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<td>$\mathcal{F}$ may be empty</td>
<td>$\mathcal{F}$ is nonempty</td>
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<td>$\mathcal{J}$ is the closure</td>
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<td>of the set of repelling</td>
<td>(see Project # 1)</td>
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<td>periodic points</td>
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A Quick Technical Note

The field $\mathbb{C}_K$ is complete, but it is usually not spherically complete.

That is, it is possible to have a decreasing chain of disks

$$D_1 \supseteq D_2 \supseteq D_3 \supseteq \cdots$$

in a (not spherically complete field) $\mathbb{C}_K$ such that

$$\bigcap_{n \geq 1} D_n = \emptyset.$$  

In this case, the disks $D_n$ must have radius bounded below by some $R > 0$.

For example, $\mathbb{C}_p$ and the Puiseux series field $\mathbb{L}$ are not spherically complete.