Chapter 22: Simultaneous Equation Models – Introduction

Chapter 22 Outline

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Chapter 22 Prep Questions

1. This question requires slogging through much high school algebra, so it is not very exciting. While tedious, it helps us understand simultaneous equation models. Consider the following two equations that model the demand and supply of beef:
Demand Model: 
\[ Q_t^D = \beta_{Const}^D + \beta_P^D P_t + \beta_{I}^D Inc_t + \epsilon_t^D \]

Supply Model: 
\[ Q_t^S = \beta_{Const}^S + \beta_P^S P_t + \beta_{Feed}^S FeedP_t + \epsilon_t^S \]

where
\[ Q_t^D = \text{Quantity of beef demanded in period } t \]
\[ Q_t^S = \text{Quantity of beef supplied in period } t \]
\[ P_t = \text{Price of beef in period } t \]
\[ Inc_t = \text{Disposable income in period } t \]
\[ FeedP_t = \text{Price of cattle feed in period } t \]

Let
\[ Q_t = \text{Equilibrium quantity in period } t : Q_t^D = Q_t^S = Q_t \]

Using algebra, solve for \( P_t \) and \( Q_t \) in terms of \( FeedP_t \) and \( Inc_t \):

a. Strategy to solve for \( P_t \):
   - Substitute \( Q_t \) for \( Q_t^D \) and \( Q_t^S \).
   - Subtract the supply model equation from the demand model equation.
   - Solve for \( P_t \).

b. Strategy to solve for \( Q_t \):
   - Substitute \( Q_t \) for \( Q_t^D \) and \( Q_t^S \).
   - Multiply the demand model equation by \( \beta_P^S \) and the supply model equation by \( \beta_P^S \).
   - Subtract the new equation for the supply model from the new equation for the demand model.
   - Solve for \( Q_t \).

2. Next, we express equations for \( P_t \) and \( Q_t \) in terms of the following \( \alpha \)'s:

   \[ Q_t = \alpha_{Const}^Q + \alpha_{FeedP}^Q FeedP_t + \alpha_{Inc}^Q Inc_t + \epsilon_t^Q \]
   \[ P_t = \alpha_{Const}^P + \alpha_{FeedP}^P FeedP_t + \alpha_{Inc}^P Inc_t + \epsilon_t^P \]

Compare these two equations for \( Q_t \) and \( P_t \) with the two equations for \( Q_t \) and \( P_t \) in problem 1. Express \( \alpha_{FeedP}^Q, \alpha_{Inc}^Q, \alpha_{FeedP}^P, \) and \( \alpha_{Inc}^P \) in terms of the \( \beta \)'s appearing in problem 1:

a. \( \alpha_{FeedP}^Q = \ldots \) \( \alpha_{Inc}^Q = \ldots \) \( \alpha_{FeedP}^P = \ldots \) \( \alpha_{Inc}^P = \ldots \)

Now, consider the following ratios of \( \alpha \)'s.

b. What does \( \frac{\alpha_{FeedP}^Q}{\alpha_{FeedP}^P} \) equal?
c. What does $\frac{\alpha^Q_i}{\alpha^P_i}$ equal?

3. In words answer the following questions:
   a. What is the goal of multiple regression analysis?
   b. What is the interpretation of each coefficient in the regression model?

Consider the following multiple regression model:

General Regression Model: $y_t = \beta_{\text{const}} + \beta_{1}x_{1t} + \beta_{2}x_{2t} + e_t$

Since the actual parameters of the model, the $\beta$'s, are unobservable, we estimate them. The estimated parameters are denoted by italicized Roman $b$'s:

$\text{Est}_y = b_{\text{const}} + b_{1}x_{1} + b_{2}x_{2}$

In terms of the estimated coefficients, $b_{x1}$ and/or $b_{x2}$, what is the expression for the estimated change in $y$?

c. If $x1$ changes by $\Delta x_1$ while $x2$ constant: $\Delta y = \ldots$
d. If $x2$ changes by $\Delta x_2$ while $x1$ constant: $\Delta y = \ldots$
e. Putting parts c and d together, if both $x1$ and $x2$ change: $\Delta y = \ldots$

4. Consider the following model of the U.S. beef market:

Demand Model: $Q^D = 100,000 - 10,000P + 150Inc$

Supply Model: $Q^S = 190,000 + 5,000P - 6,000FeedP$

Equilibrium: $Q^D = Q^S = Q$

where

- $Q$: Quantity of beef (millions of pounds)
- $P$: Real price of beef (cents per pound)
- $Inc$: Real disposable income (billions of dollars)
- $FeedP$: Real price of cattle feed (cents per pounds of corn cobs)

a. Use algebra to solve for the equilibrium price and quantity. That is,
   1) Express the equilibrium price, $P$, in terms of $FeedP$ and $Inc$.
   2) Express the equilibrium quantity, $Q$, in terms of $FeedP$ and $Inc$.
   These two equations are called the reduced form (RF) equations.

b. Suppose that $FeedP$ equals 40 and $Inc$ equals 4,000.
   1) What are the numerical values of the equilibrium price and quantity?
   2) On a sheet of graph paper, plot the demand and supply curves to illustrate the equilibrium.

c. Assume that you did not know the equations for the demand and supply models. On the other hand, you do know the reduced form (RF) equations that you derived in part a.
   1) Suppose that $Inc$ were to rise from 4,000 to 6,000 while $FeedP$ remains constant at 40. Using the reduced form (RF) equations calculate the new equilibrium price and quantity.
i. Will the demand curve shift? Explain.
ii. Will the supply curve shift? Explain.
iii. On a sheet of graph paper, plot the demand curve(s), the supply curve(s), and the two equilibria.
iv. Based on the numerical values of the two equilibria can you calculate the slope of the supply curve? Explain.
v. Based on the numerical values of the two equilibria can you calculate the price coefficient of the demand model? Explain.

2) Instead, suppose that $FeedP$ were to rise from 40 to 60 while $Inc$ remains constant at 4,000. Using the reduced form (RF) equations calculate the new equilibrium price and quantity.
   i. Will the demand curve shift? Explain.
   ii. Will the supply curve shift? Explain.
   iii. On a sheet of graph paper, plot the demand curve(s), the supply curve(s), and the two equilibria.
   iv. Based on the numerical values of the two equilibria can you calculate the price coefficient of the demand model? Explain.
   v. Based on the numerical values of the two equilibria can you calculate the price coefficient of the supply model? Explain.

**Introduction**

Demand and supply curves are arguably the economist’s most widely used tools. They provide one example of simultaneous equation models. Unfortunately, as we shall shortly show, the ordinary least squares (OLS) estimation procedure is biased when it is used to estimate the parameters of these models. To illustrate this we begin by reviewing the effect that explanatory variable/error term correlation has on the ordinary least squares (OLS) estimation procedure. Then, we focus on a demand/supply model to explain why the ordinary least squares (OLS) estimation procedure leads to bias.
Review: Explanatory Variable/Error Term Correlation

Explanatory variable/error term correlation leads to bias. When the explanatory variable and error term are positively correlated, the best fitting line is more steeply sloped than the actual equation line; consequently, the ordinary least squares (OLS) estimation procedure for the coefficient value is biased upward. On the other hand, when the explanatory variable and error term are negatively correlated, the best fitting line is less steeply sloped than the actual equation line; consequently, the ordinary least squares (OLS) estimation procedure for the coefficient value is biased downward.

Explanatory variable
and error term are
positively correlated
↓
OLS estimation procedure
for coefficient value
biased upward

Explanatory variable
and error term are
negatively correlated
↓
OLS estimation procedure
for coefficient value
biased downward
Simultaneous Equations: Demand and Supply

Consider the market for a good such as food or clothing. The following two equations describe a standard demand/supply model of the market for the good:

Demand Model: \( Q^D_t = \beta^D_{\text{const}} + \beta^D_p P_t + \text{Other Demand Factors} + e^D_t \)

Supply Model: \( Q^S_t = \beta^S_{\text{const}} + \beta^S_p P_t + \text{Other Supply Factors} + e^S_t \)

where

- \( P_t \) = Price of the good
- \( Q^D_t \) = Quantity of the good demanded
- \( e^D_t \) = Error term in the demand equation
- \( Q^S_t \) = Quantity of the good supplied
- \( e^S_t \) = Error term in the supply equation

First, focus on our notation. The superscripts denote the models:

- **Superscript D – Demand model**: \( Q^D_t \) equals the quantity demanded. \( \beta^D_{\text{const}} \), \( \beta^D_p \), and \( e^D_t \) refer to the parameters and error term of the demand model.

- **Superscript S – Supply model**: \( Q^S_t \) equals the quantity supplied. \( \beta^S_{\text{const}} \), \( \beta^S_p \), and \( e^S_t \) refer to the parameters and the error term of the supply model.

The parameter subscripts refer to the constants and explanatory variable coefficients of the models.

- **Subscript Const – Constant**: \( \beta^D_{\text{const}} \) and \( \beta^S_{\text{const}} \) are the constants of the demand model and the supply model.

- **Subscript P – Price coefficient**: \( \beta^D_p \) and \( \beta^S_p \) are the price coefficients of the demand model and the supply model.

The quantity of a good demanded is determined by the good’s own price and other demand factors such as income, the price of substitutes, the price of complements, etc. Similarly, the quantity of a good supplied is determined by the good’s own price and other supply factors such as wages, raw material prices, etc.

The market is in equilibrium whenever the quantity demanded equals the quantity supplied:

\( Q^D_t = Q^S_t = Q_t \)

Both the quantity, \( Q_t \), and the price, \( P_t \), are determined simultaneously as depicted by the famous demand/supply diagram reproduced in Figure 22.2:
In a simultaneous equation model, it is important to distinguish between endogenous and exogenous variables. Endogenous variables are variables whose values are determined “within” the model. In the demand/supply model, the quantity and the price, as determined by the intersection of the supply and demand curves, are endogenous; that is, both quantity and price are determined simultaneously “within” the model. On the other hand, exogenous variables are determined “outside” the model; in the context of the model, the values of exogenous variables are taken as given. The model does not attempt to explain how the values of exogenous variables are determined. For example, if we are considering the demand and supply models for beer, income, the price of wine, wages, the price of hops, etc. would all be exogenous variables.

- **Endogenous variables** – Variables determined “within” the model: Price and Quantity
- **Exogenous variables** – Variables determined “outside” the model: Other Factors

"Endogenous versus Exogenous Variables"

Figure 22.2: Demand/Supply Model
Single Equation versus Simultaneous Equation Models

In single equation models, there is only one endogenous variable, the dependent variable itself; all explanatory variables are exogenous. For example, in the following single model the dependent variable is consumption and the explanatory variable income:

\[ \text{Cons}_t = \beta_{\text{Const}} + \beta_{\text{Inc}t} + e_t \]

The model only attempts to explain how consumption is determined. The dependent variable, consumption, is the only endogenous variable. The model does not attempt to explain how income is determined; that is, the values of income are taken as given. All explanatory variables, in this case only income, are exogenous.

In a simultaneous equation model, while the dependent variable is endogenous, an explanatory variable can be either endogenous or exogenous. In the demand/supply model, quantity, the dependent variable, is an endogenous; quantity is determined “within” the model. Price is both an endogenous variable and an explanatory variable. Price is determined “within” the model and it is used to explain the quantity demanded and the quantity supplied.

What are the consequences of endogenous explanatory variables for the ordinary least squares (OLS) estimation procedure?

Claim: Whenever an explanatory variable is also an endogenous variable, the ordinary least squares (OLS) estimation procedure for the value of its coefficient is biased.

We shall now use the demand and supply models to justify this claim.
**Demand Model**

When the ordinary least squares (OLS) estimation procedure is used to estimate the demand model, the good’s own price and the error term are positively correlated; accordingly, the ordinary least squares (OLS) estimation procedure for the value of the price coefficient will be biased upward. Let us now show why.

Demand Model: \( Q_t^D = \beta^D_{\text{const}} + \beta^D_t P_t + \text{Other Demand Factors} + \epsilon_t^D \)

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**Ordinary Least Squares (OLS) Estimation Procedure: Our Suspicions**

When the error term, \( \epsilon_t^D \), rises the demand curve shifts to the right resulting in a higher price; on the other hand, when the error, \( \epsilon_t^D \), falls the demand curve shifts to the left resulting in a lower price. In the demand model, the price, \( P_t \), and the error term, \( \epsilon_t^D \), are positively correlated. Since the good’s own price is an explanatory variable, bias results:

\[
\begin{align*}
\epsilon_t^D \; &\text{up} \\
\downarrow \\
P_t \; &\text{up}
\end{align*}
\]

\[
\begin{align*}
\epsilon_t^D \; &\text{down} \\
\downarrow \\
P_t \; &\text{down}
\end{align*}
\]

Explanatory variable and error term positively correlated

OLS estimation procedure for coefficient value biased upward

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![Figure 22.3: Effect of the Demand Error Term](image-url)
Confirming Our Suspicions
The ordinary least squares (OLS) estimation procedure for the value of the price coefficient in the demand model should be biased upward. We shall check our logic with a simulation.

Econometrics Lab 22.1: Simultaneous Equations – Demand Price Coefficient [Link to MIT-Lab 22.1 goes here.]

Figure 22.4: Simultaneous Equation Simulation

We are focusing on the demand model; hence, the Dem radio button is selected. The lists immediately below the Dem radio button specify the demand model. The actual constant equals 30, the actual price coefficient equals \(-4\), etc. XCoef represents an “other demand factor,” such as income.
Be certain that the Pause checkbox is cleared. Click Start and then after many, many repetitions click Stop. The average of the estimated demand price coefficient values is $-2.6$, greater than the actual value, $-4.0$. This result suggests that the ordinary least squares (OLS) estimation procedure for the value of the price coefficient is biased upward. Our Econometrics Lab confirms our suspicions.

<table>
<thead>
<tr>
<th>Estimation Procedure</th>
<th>Sample Size</th>
<th>Actual Value</th>
<th>Mean (Average) of Estimated Coef Values</th>
<th>Magnitude of Bias</th>
</tr>
</thead>
<tbody>
<tr>
<td>OLS</td>
<td>20</td>
<td>$-4.0$</td>
<td>$\approx -2.6$</td>
<td>$\approx 1.4$</td>
</tr>
<tr>
<td>OLS</td>
<td>30</td>
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</tr>
<tr>
<td>OLS</td>
<td>40</td>
<td>$-4.0$</td>
<td>$\approx -2.6$</td>
<td>$\approx 1.4$</td>
</tr>
</tbody>
</table>

Table 22.1: Simultaneous Equation Simulation Results – Demand

But even though the ordinary least squares (OLS) estimation procedure is biased, it might be consistent, might it not? Recall the distinction between an unbiased and a consistent estimation procedure:

- **Unbiased**: The estimation procedure does not systematically underestimate or overestimate the actual value; that is, after many, many repetitions the average of the estimates equals the actual value.

- **Consistent but Biased**: As consistent estimation procedure can be biased. But, as the sample size, as the number of observations, grows:
  - The magnitude of the bias decreases. That is, the mean of the coefficient estimate’s probability distribution approaches the actual value.
  - The variance of the estimate’s probability distribution diminishes and approaches 0.

How can we use the simulation to investigate this possibility? Just increase the sample size. If the procedure is consistent, the average of the estimated coefficient values after many, many repetitions would move closer and closer to $-4.0$, the actual value, as we increase the sample size. That is, if the procedure is consistent, the magnitude of the bias would decrease as the sample size increases. (Also, the variance of the estimates would decrease.) So, let us increase the sample size from 20 to 30 and then to 40. Unfortunately, we observe that a larger sample size does not reduce the magnitude of the bias:

<table>
<thead>
<tr>
<th>Estimation Procedure</th>
<th>Sample Size</th>
<th>Actual Value</th>
<th>Mean (Average) of Estimated Coef Values</th>
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<td>$\approx -2.6$</td>
<td>$\approx 1.4$</td>
</tr>
</tbody>
</table>

Table 22.2: Simultaneous Equation Simulation Results – Demand
When estimating the value of the price coefficient in the demand model, the ordinary least squares (OLS) estimation procedure fails in two respects:

- **Bad news:** The ordinary least squares (OLS) estimation procedure is biased.
- **Bad news:** The ordinary least squares (OLS) estimation procedure is not consistent.

**Supply Model**

We shall now use the same line of reasoning to show that the ordinary least squares (OLS) estimation procedure for the value of the price coefficient in the supply model is also biased.

Supply Model: $Q_i^S = \beta_{\text{const}}^S + \beta_P^S P_i + \text{Other Supply Factors} + \epsilon_i^S$

Ordinary Least Squares (OLS) Estimation Procedure: Our Suspicions

When the error term, $\epsilon_i^S$, rises the supply curve shifts to the right resulting in a lower price; on the other hand, when the error term, $\epsilon_i^S$, falls the supply curve shifts to the left resulting in a higher price. In the supply model, the price, $P$, and the error term, $\epsilon_i^S$, are negatively correlated.
The ordinary least squares (OLS) estimation procedure for the value of the price coefficient in the supply model should be biased downward. Once again, we shall use a simulation to confirm our logic.

Confirming Our Suspicions

*Econometrics Lab 22.2: Simultaneous Equations – Supply Price Coefficient*

We are now focusing on the supply curve; hence, the Sup radio button is selected. Note that the actual value of the supply price coefficient equals 1.0. Be certain that the Pause checkbox is cleared. Click Start and then after many, many repetitions click Stop. The average of the estimated coefficient values is $-1.4$, less than the actual value, 1.0. This result suggests that the ordinary least squares (OLS) estimation procedure for the value of the price coefficient is biased downward, confirming our suspicions.

But might the estimation procedure be consistent? To answer this question increase the sample size from 20 to 30 and then from 30 to 40. The magnitude of the bias is unaffected. Accordingly, it appears that the ordinary least squares (OLS) estimation procedure for the value of the price coefficient is not consistent either.

<table>
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<tr>
<th>Estimation Procedure</th>
<th>Sample Size</th>
<th>Actual Coef Value</th>
<th>Mean (Average) of Estimated Coef Values</th>
<th>Magnitude of Bias</th>
</tr>
</thead>
<tbody>
<tr>
<td>OLS</td>
<td>20</td>
<td>1.0</td>
<td>$\approx-4$</td>
<td>$\approx1.4$</td>
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<tr>
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<td>30</td>
<td>1.0</td>
<td>$\approx-4$</td>
<td>$\approx1.4$</td>
</tr>
<tr>
<td>OLS</td>
<td>40</td>
<td>1.0</td>
<td>$\approx-4$</td>
<td>$\approx1.4$</td>
</tr>
</tbody>
</table>

Table 22.3: Simultaneous Equation Simulation Results – Supply
When estimating the price coefficient’s value in the supply model, the ordinary least squares (OLS) estimation procedure fails in two respects:

- **Bad news:** The ordinary least squares (OLS) estimation procedure is biased.
- **Bad news:** The ordinary least squares (OLS) estimation procedure is not consistent.

The supply model simulations illustrate a problem even worse than that encountered when estimating the demand model. In this case, the bias can be so severe that the mean of the coefficient estimate’s probability distribution has the wrong sign. To gain more intuition, suppose that the probability distribution is symmetric. Then, the chances that the coefficient estimate would have the wrong sign are greater than the chances that it would have the correct sign when using the ordinary least squares (OLS) estimation procedure. This is very troublesome, is it not?

**Summary: Endogenous Explanatory Variable Problem**

We have used the demand and supply models to illustrate that an endogenous explanatory variable creates a bias problem for the ordinary least squares (OLS) estimation procedure. Whenever an explanatory variable is also an endogenous variable, the ordinary least squares (OLS) estimation procedure for the value of its coefficient is biased.
An Example: The Market for Beef

**Beef Market Data:** Monthly time series data relating to the market for beef from 1977 to 1986.

- $Q_t$: Quantity of beef in month $t$ (millions of pounds)
- $P_t$: Real price of beef in month $t$ (1982-84 cents per pound)
- $Inc_t$: Real disposable income in month $t$ (billions of chained 2005 dollars)
- $ChickP_t$: Real price of whole chickens in month $t$ (1982-84 cents per pound)
- $FeedP_t$: Real price of cattle feed in month $t$ (1982-84 cents per pounds of corn cobs)

**Demand and Supply Models**

We begin by describing the endogenous and exogenous variables:

- **Endogenous Variables:** Both the quantity of beef and the price of beef, $Q_t$ and $P_t$, are endogenous variables; they are determined within the model.

- **Exogenous Variables:**
  - Disposable income is an “Other Demand Factor”; disposable income, $Inc_t$, is an exogenous variable that affects demand. Since beef is regarded as a normal good, households demand more beef when income rises.
  - The price of chicken is also an “Other Demand Factor”; the price of chicken, $ChickP_t$, is an exogenous variable that affects demand. Since chicken is a substitute for beef, households demand more beef when the price of chicken rises.
  - The price of cattle feed is an “Other Supply Factor”; the price of cattle feed, $FeedP_t$, is an exogenous variable that affects supply. Since cattle feed is an input to the production of beef, firms produce less when the price of cattle feed rises.

Now let us formalize the simultaneous demand/supply model that we shall investigate:

**Demand Model:** $Q_t^D = \beta^D_{Const} + \beta^D_P P_t + \beta^D_{Inc} Inc_t + \epsilon_t^D$

**Supply Model:** $Q_t^S = \beta^S_{Const} + \beta^S_P P_t + \beta^S_{FeedP} FeedP_t + \epsilon_t^S$

**Equilibrium:** $Q_t^D = Q_t^S = Q_t$

**Endogenous Variables:** $Q_t$ and $P_t$

**Exogenous Variables:** $FeedP_t$ and $Inc_t$

**Project:** Estimate the beef market demand and supply parameters
Ordinary Least Squares (OLS) Estimation Procedure

Let us begin by using the ordinary least squares (OLS) procedure to estimate the parameters:

Ordinary Least Squares (OLS)

Dependent Variable: \( Q \)
Explanatory Variable(s): Estimate SE \( t \)-Statistic Prob
\( P \) \(-364.3646\) 18.29792 \(-19.91290\) 0.0000
\( Inc \) \(23.74785\) 0.929082 \(25.56056\) 0.0000
\( Const \) \(155137.0\) 4731.400 \(32.78882\) 0.0000

Number of Observations 120

Estimated Equation: \( \text{Est}Q^D = 155,137 - 364.4P + 23.75Inc \)

Interpretation of Estimates:
\( b_P^D = -364.4: \) A 1 cent increase in the price of beef decreases the quantity demanded by 364.4 million pounds.
\( b_{Inc}^D = 23.75: \) A 1 billion dollar increase in real disposable income increases the quantity of beef demanded by 23.75 million pounds.

Table 22.4: OLS Regression Results – Demand Model

As reported in Table 22.4, the estimate of the demand model’s price coefficient is negative, \(-364.4\), suggesting that higher prices decrease the quantity demanded. The result is consistent with economic theory suggesting that the demand curve is downward sloping.
Ordinary Least Squares (OLS)

Dependent Variable: $Q$

Explanatory Variable(s):

<table>
<thead>
<tr>
<th></th>
<th>Estimate</th>
<th>SE</th>
<th>t-Statistic</th>
<th>Prob</th>
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</thead>
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<tr>
<td>$P$</td>
<td>-231.5487</td>
<td>41.19843</td>
<td>-5.620328</td>
<td>0.0000</td>
</tr>
<tr>
<td>FeedP</td>
<td>-700.3695</td>
<td>119.3941</td>
<td>-5.866031</td>
<td>0.0000</td>
</tr>
<tr>
<td>Const</td>
<td>272042.7</td>
<td>7793.872</td>
<td>34.90469</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Number of Observations 120

Estimated Equation: $\hat{Q} = 272,043 - 231.5P + 700.1FeedP$

Interpretation of Estimates:

$b^S_P = -231.5$: A 1 cent increase in the price of beef decreases the quantity supplied by 231.5 million pounds.

$b^S_{FeedP} = -700.1$: A 1 cent increase in cattle feed decreases the quantity of beef supplied by 700.1 million pounds.

As reported in Table 22.5, the estimate of the supply model’s price coefficient is negative, $b_p = -231.5$, suggesting that higher prices decrease the quantity supplied. Obviously, this result is not consistent with economic theory. This result suggests that the supply curve is downward sloping rather than upward sloping.

But what have we just learned about the ordinary least squares (OLS) estimation procedure. The ordinary least squares (OLS) estimation procedure for the price coefficient estimate of the supply model will be biased downward. This could explain our result, could it not?

Reduced Form (RF) Estimation Procedure: The Mechanics

We shall now describe an alternative estimation procedure, the reduced form (RF) estimation procedure. We shall show that while this new procedure does not “solve” the bias problem, it mitigates it. More specifically, while the procedure is still biased, it proves to be consistent. In this way, the new procedure is “better than” ordinary least squares (OLS). We begin by describing the mechanics of the reduced form (RF) estimation procedure.
We have argued that the ordinary least squares (OLS) estimation procedure leads to bias because an endogenous variable, in our case the price, is an explanatory variable. The reduced form (RF) approach begins by using algebra to express each endogenous variable only in terms of the exogenous variables. These new equations are called the reduced form (RF) equations.

**Intuition:** Since bias results from endogenous explanatory variables, algebraically manipulate the simultaneous equation model to express each endogenous variable only in terms of the exogenous variables. Then, use the ordinary least squares (OLS) estimation procedure to estimate the parameters of these newly derived equations, rather than the original ones.

The reduced form (RF) approach involves three steps:

- **Step 1:** Derive the Reduced Form (RF) Equations from the Original Models.
  - The reduced form (RF) equations express each endogenous variable in terms of the exogenous variables only.
  - Algebraically solve for the original model’s parameters in terms of the reduced form (RF) parameters.
- **Step 2:** Use Ordinary Least Squares (OLS) Estimation Procedure to Estimate the Parameters of the Reduced Form (RF) Equations.
- **Step 3:** Calculate Coefficient Estimates for the Original Models Using the Derivations from Step 1 and Estimates from Step 2.

**Step 1:** Derive the Reduced Form (RF) Equations from the Original Models

We begin with the supply and demand models:

**Demand Model:**

\[ Q^D_t = \beta^D_{\text{Const}} + \beta^D_p P_t + \beta^D_I \text{Inc}_t + e^D_t \]

**Supply Model:**

\[ Q^S_t = \beta^S_{\text{Const}} + \beta^S_p P_t + \beta^S_{\text{Feed}} \text{Feed}_t + e^S_t \]

**Equilibrium:**

\[ Q^D_t = Q^S_t = Q_t \]

Endogenous Variables: \( Q_t \) and \( P_t \)
Exogenous Variables: \( \text{Feed}_t \) and \( \text{Inc}_t \)

There are six parameters of the demand and supply models: \( \beta^D_{\text{Const}} \), \( \beta^D_p \), \( \beta^D_I \), \( \beta^S_{\text{Const}} \), \( \beta^S_p \), and \( \beta^S_{\text{Feed}} \). We wish to estimate the values of these parameters.
The reduced form (RF) equations express each endogenous variable in terms of the exogenous variables. In this case, we wish to express $Q_t$ in terms of $FeedPt_t$ and $Inc_t$, and $P_t$ in terms of $FeedPt_t$ and $Inc_t$. Appendix 22.1 shows that how elementary, yet laborious, algebra can be used to derive the following reduced form (RF) equations for the endogenous variables, $Q_t$ and $P_t$:

$$Q_t = \frac{\beta^D_P \beta^S_{Const} - \beta^D_P \beta^S_{Const}}{\beta^P_P - \beta^D_P} - \frac{\beta^S_P \beta^D_{PP}}{\beta^P_P - \beta^D_P} FeedP_t + \frac{\beta^S_P \beta^D_{tI}}{\beta^P_P - \beta^D_P} Inc_t + \frac{\beta^S_P \epsilon^D_t - \beta^D_P \epsilon^S_t}{\beta^P_P - \beta^D_P}$$

$$P_t = \frac{\beta^S_P \beta^D_{Const} - \beta^S_P \beta^D_{Const}}{\beta^P_P - \beta^D_P} - \frac{\beta^S_P \beta^D_{PP}}{\beta^P_P - \beta^D_P} FeedP_t + \frac{\beta^S_P \beta^D_{tI}}{\beta^P_P - \beta^D_P} Inc_t + \frac{\epsilon^D_t - \epsilon^S_t}{\beta^P_P - \beta^D_P}$$

Now, let us make an interesting observation about the reduced form (RF) equations. Focus first on the ratio of the feed price coefficients and then on the ratio of the income coefficients. These ratios equal the price coefficients of the original demand and supply models, $\beta^D_P$ and $\beta^S_P$:

<table>
<thead>
<tr>
<th>Feed Price Coefficients in the Reduced Form (RF) Equations</th>
<th>Income Coefficients in the Reduced Form (RF) Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio of Feed Price Coefficients Equals the Price Coefficient of the Demand Model</td>
<td></td>
</tr>
<tr>
<td>$\frac{\beta^D_P \beta^S_{FP}}{\beta^P_P - \beta^D_P} = \beta^D_P$</td>
<td></td>
</tr>
<tr>
<td>$\frac{\beta^S_P \beta^D_{PP}}{\beta^P_P - \beta^D_P} = \beta^S_P$</td>
<td></td>
</tr>
</tbody>
</table>

We shall formalize this observation by expressing all the parameters of the original simultaneous equation model (the constants and coefficients of the demand and supply models, the $\beta$'s) in terms of the parameters of the reduced form (RF) model (the constants and coefficients of the reduced form equations). To do so, let $\alpha$ represent the parameters, the constants and coefficients, of the reduced form (RF) equations:

$$Q_t = \alpha^Q_{Const} + \alpha^Q_{FP} FeedP_t + \alpha^Q_{tI} Inc_t + \epsilon^Q_t$$

$$P_t = \alpha^P_{Const} + \alpha^P_{FP} FeedP_t + \alpha^P_{tI} Inc_t + \epsilon^P_t$$
First, consider the notation we use in the reduced form (RF) equations. Superscripts refer to the reduced form (RF) equation:

- **Superscript Q – Quantity Reduced Form (RF) Equation:** $\alpha_{\text{Const}}^Q$, $\alpha_{FP}^Q$, $\alpha_I^Q$, and $\epsilon_i^Q$ are the parameters and error term of the quantity reduced form (RF) equation.

- **Superscript P – Price Reduced Form (RF) Equation:** $\alpha_{\text{Const}}^P$, $\alpha_{FP}^P$, $\alpha_I^P$, and $\epsilon_i^P$ are the parameters and the error term of the price reduced form (RF) equation.

The parameter subscripts refer to the constants and coefficients of each reduced form (RF) equation:

- **Subscript Const – Reduced Form (RF) Constants:** $\alpha_{\text{Const}}^Q$ and $\alpha_{\text{Const}}^P$ are the constants of the quantity and price reduced form (RF) equations.

- **Subscript FP – Reduced Form (RF) Feed Price Coefficients:** $\alpha_{FP}^Q$ and $\alpha_{FP}^P$ are the feed price coefficients of the quantity and price reduced form (RF) equations.

- **Subscript I – Reduced Form (RF) Income Coefficients:** $\alpha_I^Q$ and $\alpha_I^P$ are the income coefficients of the quantity and price reduced form (RF) equations.

There are 6 parameters of the reduced form (RF) equations: $\alpha_{\text{Const}}^Q$, $\alpha_{FP}^Q$, $\alpha_I^Q$, $\alpha_{\text{Const}}^P$, $\alpha_{FP}^P$, and $\alpha_I^P$. 
By comparing the two sets of reduced form (RF) equations we can express each of the reduced form (RF) parameter, each \( \alpha \), in terms of the parameters of the original demand and supply models, the \( \beta \)'s. We have six equations:

\[
\alpha^Q_{\text{Const}} = \frac{\beta^D_p \beta^D_{\text{Const}} - \beta^D_p \beta^S_{\text{Const}}}{\beta^D_p - \beta^D_D} \\
\alpha^Q_F = -\frac{\beta^D_p \beta^S_F}{\beta^D_p - \beta^D_D} \\
\alpha^L = \frac{\beta^S_p \beta^D_I}{\beta^D_p - \beta^D_D} \\
\alpha^P_{\text{Const}} = \frac{\beta^S_{\text{Const}} - \beta^S_F}{\beta^S_p - \beta^D_D} \\
\alpha^P_F = -\frac{\beta^S_F}{\beta^D_p - \beta^D_D} \\
\alpha^P_I = \frac{\beta^D_I}{\beta^D_p - \beta^D_D}
\]

There are six parameters of the original demand and supply models: \( \beta^D_{\text{Const}}, \beta^D_p, \beta^D_I, \beta^S_{\text{Const}}, \beta^S_p, \text{ and } \beta^S_F \). That is, we have six unknowns, the \( \beta \)'s. We have six equations and six unknowns. We can solve for the unknowns by expressing the \( \beta \)'s in terms of the \( \alpha \)'s. For example, we can solve for price coefficients of the original demand and supply models, \( \beta^D_p \) and \( \beta^S_p \):

Feed Price Coefficients in the Reduced Form (RF) Equations:

\[
\frac{\alpha^Q_F}{\alpha^P_F} = \frac{\beta^D_p}{\beta^S_p - \beta^D_p} = \beta^D_p \\
\frac{\alpha^Q_I}{\alpha^P_I} = \frac{\beta^D_I}{\beta^D_p - \beta^D_D} = \beta^S_p
\]

Income Coefficients in the Reduced Form (RF) Equations:

\[
\frac{\alpha^Q_F}{\alpha^P_F} = \frac{\beta^D_p \beta^D_I}{\beta^D_p - \beta^D_D} = \beta^D_p \\
\frac{\alpha^Q_I}{\alpha^P_I} = \frac{\beta^S_p \beta^D_I}{\beta^D_p - \beta^D_D} = \beta^S_p
\]

These coefficients reflect the “slopes” of the demand and supply curves.
Step 2: Use Ordinary Least Squares (OLS) to Estimate the Reduced Form Equations
We use the ordinary least squares (OLS) estimation procedure to estimate the $\alpha$'s:

Ordinary Least Squares (OLS)

Dependent Variable: $Q$

<table>
<thead>
<tr>
<th>Explanatory Variable(s)</th>
<th>Estimate</th>
<th>SE</th>
<th>$t$-Statistic</th>
<th>Prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>FeedP</td>
<td>-331.9966</td>
<td>121.6865</td>
<td>-2.728293</td>
<td>0.0073</td>
</tr>
<tr>
<td>Inc</td>
<td>17.34683</td>
<td>2.132027</td>
<td>8.136309</td>
<td>0.0000</td>
</tr>
<tr>
<td>Const</td>
<td>138725.5</td>
<td>13186.01</td>
<td>10.52066</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Number of Observations 120

Estimated Equation: $\hat{Q} = 138,726 - 332.00 FeedP + 17.347 Inc$

Interpretation of Estimates:
- $a_{\text{FeedP}}^Q = -332.00$: A 1 cent increase in the price of cattle feed decreases the quantity of beef by 332.00 million pounds.
- $a_{\text{Inc}}^Q = 17.347$: A 1 billion dollar increase in real disposable income increases the quantity of beef by 17.347 million pounds.

Table 22.6: OLS Regression Results – Quantity Reduced Form (RF) Equation

Ordinary Least Squares (OLS)

Dependent Variable: $P$

<table>
<thead>
<tr>
<th>Explanatory Variable(s)</th>
<th>Estimate</th>
<th>SE</th>
<th>$t$-Statistic</th>
<th>Prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>FeedP</td>
<td>1.056242</td>
<td>0.286474</td>
<td>3.687044</td>
<td>0.0003</td>
</tr>
<tr>
<td>Inc</td>
<td>0.018825</td>
<td>0.005019</td>
<td>3.750636</td>
<td>0.0003</td>
</tr>
<tr>
<td>Const</td>
<td>33.02715</td>
<td>31.04243</td>
<td>1.063936</td>
<td>0.2895</td>
</tr>
</tbody>
</table>

Number of Observations 120

Estimated Equation: $\hat{P} = 33.037 + 1.0562 FeedP + .018825 Inc$

Interpretation of Estimates:
- $a_{\text{FeedP}}^P = 1.0562$: A 1 cent increase in the price of cattle feed increases the price of beef by 1.0562 cents.
- $a_{\text{Inc}}^P = 17.347$: A 1 billion dollar increase in real disposable income increases the price of beef by .018825 cents.

Table 22.7: OLS Regression Results – Price Reduced Form (RF) Equation
Table 22.8 summarizes the reduced form (RF) coefficient estimates:

<table>
<thead>
<tr>
<th></th>
<th>Quantity Reduced Form (RF) Coefficient Estimate</th>
<th>Price Reduced Form (RF) Coefficient Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Feed Price</td>
<td>−332.00</td>
<td>1.0562</td>
</tr>
<tr>
<td>Income</td>
<td>17.347</td>
<td>0.018825</td>
</tr>
</tbody>
</table>

Table 22.8: Reduced Form (RF) Coefficient Estimates

Let us now take a brief aside to show that the reduced form (RF) estimates are consistent with the standard demand/supply analysis for the beef market.

- **Feed Price Reduced Form (RF) Estimates**: Since cattle feed is an input for beef production, an increase in the feed price shifts the supply curve for beef to the left. As Figure 22.7 illustrates, the equilibrium quantity falls and the equilibrium price rises.

![Diagram of Demand/Supply Analysis – An Increase in Feed Price](image)

The feed price coefficient estimate in the quantity reduced form (RF) equation is negative, −332.00. The negative estimate suggests that an increase in feed prices reduces the quantity. The feed price estimate in the price reduced form (RF) equation is positive, 1.0562. This suggests that an increase in the feed price increases the price of beef.

The feed price coefficient estimates are consistent with the standard demand/supply analysis.
• **Income Reduced Form (RF) Estimates:** Since beef is generally regarded as a normal good, an increase in income shifts the demand curve for beef to the right. As Figure 22.8 illustrates, the equilibrium quantity and price both increase.

![Diagram of Demand/Supply Analysis – An Increase in Income](image)

The income coefficient estimates in both the quantity and price reduced form (RF) regression are positive, 17.347 and .018825. The positive estimates suggest that an increase in income cause both the quantity and the price of beef to rise.

- Income increases
- Quantity rises
- Price falls

The income coefficient estimates are consistent with the standard demand/supply analysis.

We now return to complete Step 3 of the reduced form (RF) estimation procedure.
Step 3: Calculate Coefficient Estimates for the Original Model Using the Derivations and Estimates from Steps 1 and 2. We shall use the reduced form (RF) coefficient estimates from Step 2 to estimate the price coefficients of the demand and supply models, $\beta_p^D$ and $\beta_p^S$, the “slopes” of the supply and demand curves. To do so we apply the equations for $\beta_p^D$ and $\beta_p^S$ that we derived in Step 1.

$$\beta_p^D = \frac{a_{FP}^D}{a_{FP}^P}$$

$$\beta_p^S = \frac{a_{FP}^S}{a_{FP}^P}$$

We estimate

$\beta_p^D = b_p^D = \frac{a_{FP}^D}{a_{FP}^P}$

$\beta_p^S = b_p^S = \frac{a_{FP}^S}{a_{FP}^P}$

From Step 2:

$\alpha_{FP}^D = -332.00$

$\alpha_{FP}^S = 17.347$

$\alpha_{FP}^P = 1.0562$

$\alpha_{FP}^S = .018825$

$$b_p^D = \frac{a_{FP}^D}{a_{FP}^P} = -332.00$$

$$b_p^S = \frac{a_{FP}^S}{a_{FP}^P} = 17.347$$

Comparing $b_p^D$ and $b_p^S$

$-332.00 < 17.347$

Comparing Ordinary Least Squares (OLS) and Reduced Form (RF) Estimates

We shall now compare the ordinary least squares (OLS) and reduced form (RF) estimates of the price coefficients:

<table>
<thead>
<tr>
<th>Estimated Demand Curve Price Coefficient</th>
<th>Estimated Supply Curve Price Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_p^D$</td>
<td>$b_p^S$</td>
</tr>
<tr>
<td>Ordinary Least Squares (OLS)</td>
<td>$-364.4$</td>
</tr>
<tr>
<td>Reduced form (RF)</td>
<td>$-314.3$</td>
</tr>
<tr>
<td></td>
<td>$921.5$</td>
</tr>
</tbody>
</table>

Table 22.9: Comparing OLS and RF Price Coefficient Estimates

The supply curve price coefficient is the most obvious difference. The ordinary least squares (OLS) estimate is negative while the reduced form (RF) estimate is positive. In view of our upward sloping supply curve theory, this result is comforting. Unlike the ordinary least squares (OLS) estimates, the signs of the
reduced form (RF) price coefficient estimates are consistent not only with our theory of demand, but also our theory of supply. Consequently, we would like to claim that the reduced form (RF) estimation procedure is “better” than the ordinary least squares (OLS) estimation procedure when estimating simultaneous equation models. We shall now use our Econometric lab to justify this claim.

**Justifying the Reduced Form (RF) Estimation Procedure**

Previously, we used the Simultaneous Equation simulation to show that the ordinary least squares (OLS) estimation procedure was neither unbiased nor consistent when estimating the values of the price coefficients. Now, we shall use this simulation to investigate the properties of the reduced form (RF) estimation procedure. It would be wonderful if the reduced form (RF) approach were unbiased. Failing that, might the reduced form (RF) approach be consistent? While we could address these issues rigorously, we shall avoid the complex mathematics by using a simulation.

*Econometrics Lab 22.3: Reduced From (RF) Estimation Procedure*

Note that the reduced form (RF), rather than the ordinary least squares (OLS), estimation procedure is now selected. Also, the Dem radio button is selected initially; the demand model is being analyzed. Be certain that the Pause checkbox is cleared. Click Start and then after many, many repetitions click Stop. Then, select the Sup radio button and repeat the process to analyze the supply model.

<table>
<thead>
<tr>
<th>Estimation Procedure</th>
<th>Sample Size</th>
<th>Demand Price Coefficient</th>
<th>Supply Price Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sample Size</td>
<td>Actual Coef Value</td>
<td>Mean of Estimated Coef Values</td>
</tr>
<tr>
<td>RF</td>
<td>20</td>
<td>−4.0</td>
<td>≈−4.3</td>
</tr>
</tbody>
</table>

*Table 22.10: Simultaneous Equation Simulation Results*

Table 22.10 reports the reduced form (RF) results for a sample size of 20. The results suggest that reduced form (RF) estimation procedures for the price coefficients are biased. The averages of the estimated price coefficient values after many, many repetitions do not equal the actual values for either the demand or supply models. The average of the demand price coefficient estimates equals −4.3 while the actual value equals −4.0.
But perhaps that unlike the ordinary least squared (OLS) estimation procedure, the reduced form (RF) approach might be consistent. To address this question we increase the sample size, first from 20 to 30 and then from 30 to 40:

<table>
<thead>
<tr>
<th>Estimation Procedure</th>
<th>Sample Size</th>
<th>Demand Coefficient</th>
<th>Supply Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Actual Coef</td>
<td>Mean of Estimated Coef Values</td>
</tr>
<tr>
<td>RF</td>
<td>20</td>
<td>−4.0</td>
<td>≈−4.3</td>
</tr>
<tr>
<td>RF</td>
<td>30</td>
<td>−4.0</td>
<td>≈−4.2</td>
</tr>
<tr>
<td>RF</td>
<td>40</td>
<td>−4.0</td>
<td>≈−4.1</td>
</tr>
</tbody>
</table>

Table 22.11: Simultaneous Equation Simulation Results

As the sample size becomes larger, bias is still present, but the magnitude of the bias diminishes. Furthermore, the variance also falls. The simulation illustrates that while the reduced form (RF) estimation procedure for the coefficient value is still biased, it is consistent.

Table 22.11 also reports on the supply model. When the sample size equals 20, the average of the supply price coefficient estimates equals 1.3 while the actual value equals 1.0. The reduced form (RF) estimation procedure for the coefficient value is still biased. When we increase the sample size from 20 to 30 and then from 30 to 40, both the magnitude to the bias and the variance diminish suggesting that the reduced form (RF) estimation procedure for the coefficient value is consistent.

Conclusion: The reduced form (RF) estimation procedure for the coefficient value of an endogenous explanatory variable provides both good news and bad news:

- **Bad news**: The reduced form (RF) estimation procedure for the coefficient value is biased.
- **Good news**: The reduced form (RF) estimation procedure for the coefficient value is consistent.
Two Paradoxes

Let us reexamine how we obtained the estimates for the price coefficients of the demand and supply models:

\[
b_D = \frac{a_{FP}^0}{a_{FP}^p} = \frac{-332.00}{1.0562} = -314.3 \quad b_S = \frac{a_I^0}{a_I^p} = \frac{17.347}{0.018825} = 921.5
\]

These equations for the two price coefficient estimates appear paradoxical at first glance:

- The demand model’s price coefficient, \( b_D \), depends on the reduced form (RF) coefficients of feed price, \( a_{FP}^0 \) and \( a_{FP}^p \). But \( a_{FP}^0 \) and \( a_{FP}^p \) tell us something about supply, not demand. They tell us how the feed price, a variable that shifts the supply curve, affects the equilibrium quantity and price.
- Similarly, the supply model’s price coefficient, \( b_S \), depends on the reduced form (RF) coefficients of income, \( a_I^0 \) and \( a_I^p \). But \( a_I^0 \) and \( a_I^p \) tell us something about demand, not supply. They tell us how income, a variable that shifts the demand curve, affects the equilibrium quantity and price.

We shall resolve these paradoxes by:

- Reviewing the goal of multiple regression analysis and the interpretation of the coefficient estimates.
- Applying the interpretation of the coefficient estimates:
  - First, to the original simultaneous equation models.
  - Second, to the reduced form (RF) equations.

Resolving the Paradoxes: Coefficient Interpretation Approach

Review: Goal of Multiple Regression Analysis and the Interpretation of the Coefficients

Now let us review two key concepts:

- **Goal of Multiple Regression Analysis**: Multiple regression analysis attempts to sort out the individual effect that each explanatory variable has on the dependent variable.
- **Interpretation of Coefficients**: Each explanatory variable’s coefficient reveals the individual impact which that explanatory variable has on the dependent variable; that is, each explanatory variable’s coefficient tells us how changes in that explanatory variable affect the dependent variable while all other explanatory variables remain constant.
Consider a general regression model with two explanatory variables:

\[ y_t = \beta_{\text{const}} + \beta_{x1}x_{1t} + \beta_{x2}x_{2t} + e_t \]

where
- \( y_t \) = Value of the dependent variable for observation \( t \)
- \( x_{1t} \) = Value of the explanatory variable 1 for observation \( t \)
- \( x_{2t} \) = Value of the explanatory variable 2 for observation \( t \)

Since the actual parameters of the model, the \( \beta \)'s are unobservable, we estimate them:

\[ \hat{y}_t = b_{\text{const}} + b_{x1}x_{1t} + b_{x2}x_{2t} \]

The coefficient estimates attempt to separate out the individual effect that each explanatory variable has on the dependent variable. To explain what this means, focus on the estimate of the first explanatory variable’s coefficient, \( b_{x1} \). It estimates the change in the dependent variable resulting from a change in the explanatory variable 1 while all other explanatory variables remain constant.

More formally,

\[ \Delta y = b_{x1}\Delta x_1 \quad \text{or} \quad b_{x1} = \frac{\Delta y}{\Delta x_1} \quad \text{while all other explanatory variables remain constant} \]

where
- \( \Delta x_1 \) = Change in explanatory variable 1
- \( \Delta y \) = Estimated change in dependent variable

A little algebra explains why. We begin with the equation estimating our model:

\[ \hat{y}_t = b_{\text{const}} + b_{x1}x_{1t} + b_{x2}x_{2t} \]

Now, increase the explanatory variable 1 by \( \Delta x_1 \) while keeping all other explanatory variables constant. \( \Delta y \) estimates the resulting change in the dependent variable.

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price: ( x_1 )</td>
<td>( x_1 + \Delta x_1 )</td>
</tr>
<tr>
<td>Quantity: ( \hat{y}_t )</td>
<td>( \hat{y}_t + \Delta y )</td>
</tr>
</tbody>
</table>

\[ \Delta y = b_{x1}\Delta x_1 \]

or

\[ b_{x1} = \frac{\Delta y}{\Delta x_1} \]
In the equation estimating our model, substitute 
\( \text{Esty} + \Delta y \) for \( \text{Esty} \) and 
\( x_1 + \Delta x_1 \) for \( x_1 \)

\[
\text{Esty} = b_{\text{Const}} + b_{x_1} x_1 + b_{x_2} x_2 \\
\downarrow \\
\text{Esty} + \Delta y = b_{\text{Const}} + b_{x_1} (x_1 + \Delta x_1) + b_{x_2} x_2 \\
\downarrow \\
\text{Esty} + \Delta y = b_{\text{Const}} + b_{x_1} x_1 + b_{x_1} \Delta x_1 + b_{x_2} x_2
\]

Substituting

\[
\text{Esty} = b_{\text{Const}} + b_{x_1} x_1 + b_{x_2} x_2 \\
\downarrow \\
\Delta y = 0 + b_{x_1} \Delta x_1 + 0
\]

Multiplying through by \( b_{x_1} \)

Original equation

Subtracting the equations

Simplifying

Dividing through by \( \Delta x_1 \)

\[
\frac{\Delta y}{\Delta x_1} = b_{x_1} \text{ while all other explanatory variables constant}
\]

Using the same logic we can interpret the estimate of the second explanatory variable’s coefficient, \( b_{x_2} \), analogously:

\[
\Delta y = b_{x_2} \Delta x_2 \quad \text{or} \quad b_{x_2} = \frac{\Delta y}{\Delta x_2} \text{ while all other explanatory variables remain constant}
\]

\( b_{x_2} \) allows us to estimate the change in the dependent variable when explanatory variable 2 changes while all other explanatory variables remain constant.

What happens when both explanatory variables change simultaneously?

The total estimated change in the quantity demanded equals the sum of the individual changes:

\[
\Delta y = b_{x_1} \Delta x_1 + b_{x_2} \Delta x_2
\]

Each term estimates the change in the dependent variable resulting from a change in each individual explanatory variable. We shall now apply the interpretation of the coefficient estimates to resolve the paradoxes.
Paradox: Demand Model Price Coefficient Depend on the Reduced Form (RF) Feed Price Coefficients

We shall first explain why the price coefficient estimate of the demand model, \( b_p^D \), is determined by the reduced form (RF) feed price coefficient estimates, \( a_{FP}^p \) and \( a_{FP}^{p'} \).

Recall the demand model:

\[
\text{Demand Model: } Q_t^D = \beta_{\text{Const}}^D + \beta_p^D P_t + \beta_i^D Inc_t + e_t^D
\]

The following equation estimates the quantity demanded:

\[
\text{Est} Q^D = b_{\text{Const}}^D + b_p^D P + b_i^D Inc
\]

Interpret the price coefficient estimate, \( b_p^D \). The price coefficient estimate of the demand model estimates the change in the quantity of beef demanded when price of beef changes while income remains constant.

\[
\Delta Q^D = b_p^D \Delta P \quad \text{while income remains constant}
\]

Solving for \( b_p^D \):

\[
b_p^D = \frac{\Delta Q^D}{\Delta P} \quad \text{while income remains constant}
\]

Since income remains constant, the demand curve does not shift; hence, \( b_p^D \) is just the estimated “slope” of the demand curve for beef.
Next, consider the reduced form (RF) equations that estimate the quantity and price:

\[
\begin{align*}
\text{Est}Q &= 138,726 - 332.00\text{FeedP} + 17.347\text{Inc} \\
\text{Est}P &= 33.027 + 1.0562\text{FeedP} + .018825\text{Inc}
\end{align*}
\]

Suppose that the feed price decreases while income remains constant. As shown in Figure 22.10, the decrease in feed prices shifts the supply curve for beef to the right.

![Figure 22.10: Feed Price Decreases and Income Remains Constant](image)

Now, interpret the feed price coefficients in the reduced form (RF) equations:

- The feed price coefficient of the quantity reduced form (RF) equation estimates the change in the beef quantity of beef when the feed price changes while income remains constant:
  \[
  \Delta Q = -332.00 \Delta \text{FeedP} \quad \text{while income remains constant}
  \]

- The feed price coefficient of the quantity reduced form (RF) equation estimates the change in the beef price when the feed price changes while income remains constant:
  \[
  \Delta P = 1.0562 \Delta \text{FeedP} \quad \text{while income remains constant}
  \]

Divide \( \Delta Q \) by \( \Delta P \). While income remains constant:

\[
\frac{\Delta Q}{\Delta P} = \frac{-332.00 \Delta \text{FeedP}}{1.0562 \Delta \text{FeedP}} = -314.3 \quad \text{while income remains constant}
\]
Next, recognize that $Q$ represents the quantity of beef demanded. The change in the feed price causes the supply curve to shift, but the demand curve remains stationary because income has remained constant. As Figure 22.10 illustrates, we are moving from one point on the demand curve to another point on the same demand curve. This movement represents a change in the quantity of beef demanded, $Q^D$:

$$\frac{\Delta Q^D}{\Delta P} = -314.3$$

We now can appreciate why the “slope” of the demand curve for beef is estimated by the reduced form (RF) feed price coefficients. Changes in the feed price cause the supply curve for beef to shift. When the demand curve remains stationary, changes in the feed price move the equilibrium from one point on the demand curve to another point on the same demand curve. Consequently, the feed price coefficients of the reduced form (RF) equations estimate how the quantity and price change as we move along the demand curve because they are based on the premise that income remains constant and therefore the demand curve remains stationary. The reduced form (RF) feed price coefficients provide us with the information we need to calculate the “slope” of the demand curve for beef.

Paradox: Supply Model Price Coefficient Depend on the Reduced Form (RF) Income Coefficients

We shall use similar logic to explain why is the price coefficient estimate of the supply model, $\beta_p^S$, is determined by the reduced form (RF) income coefficient estimates, $a_t^O$ and $a_t^P$. Recall the supply model:

Supply Model: $Q_t^S = \beta_{const} + \beta_P^P P_t + \beta_{Feed}^P FeedP_t + \epsilon_t^S$

The following equation estimates the quantity supplied:

Begin by focusing on the supply model estimated equation:

$EstQ_t^S = b_{const}^S + b_P^S P + b_{Feed}^S FeedP$

Interpret the price coefficient estimate, $b_P^S$. The price coefficient estimate of the supply model estimates the change in the quantity of beef supplied when price of beef changes while the feed price remains constant:

$$\Delta Q^S = b_P^S \Delta P \quad \text{while feed price remains constant}$$

Solving for $b_P^S$:

$$b_P^S = \frac{\Delta Q^S}{\Delta P} \quad \text{while feed price remains constant}$$
Since the feed price is constant, the supply curve does not shift; hence, \( b_p^S \) is just the estimated “slope” of the supply curve for beef.

Once again, consider the reduced form (RF) equations that estimate the quantity and price:

\[
\begin{align*}
\text{Est}Q &= 138,726 - 332.00\text{FeedP} + 17.347\text{Inc} \\
\text{Est}P &= 33.027 + 1.0562\text{FeedP} + .018825\text{Inc}
\end{align*}
\]

Suppose that income increases and feed price remains constant. As shown in Figure 22.12, the demand curve will shift to the right. Now, interpret the income coefficients in the reduced form (RF) equations:

- The income coefficient of the quantity reduced form (RF) equation estimates the change in beef quantity when income changes while feed prices remain constant:
  \[
  \Delta Q = 17.347\Delta Inc \quad \text{while feed price remains constant}
  \]

- The income coefficient of the price reduced form (RF) equation estimates the change in the beef price changes income changes while feed prices remain constant:
  \[
  \Delta P = .018825\Delta Inc \quad \text{while feed price remains constant}
  \]
Dividing $\Delta Q$ by $\Delta P$, we have
\[ \frac{\Delta Q}{\Delta P} = \frac{17.347\Delta \text{Inc}}{018825\Delta \text{Inc}} = \frac{17.347}{018825} = 921.5 \]
while feed price remains constant.

Next, recognize that this $Q$ represents the quantity of beef supplied. The change in income causes the demand curve to shift, but the supply curve remains stationary because the feed price has remained constant. As Figure 22.12 illustrates, we move from one point on the supply curve to another point on the same supply curve. This movement represents a change in the quantity of beef supplied, $Q^S$:
\[ \frac{\Delta Q^S}{\Delta P} = 921.5 \]

We can appreciate why the “slope” of the supply curve for beef is determined by the reduced form (RF) income coefficients. Changes in income cause the demand curve for beef to shift. When the supply curve remains stationary, changes in income move the equilibrium from one point on the supply curve to another point on the same supply curve. Consequently, the income coefficients of the reduced form (RF) equations estimate how the quantity and price change as we move along the supply curve because they are based on the premise that the feed price remains constant and therefore the supply curve remains stationary. The reduced form (RF) income coefficients provide us with the information we need to calculate the “slope” of the supply curve for beef.
The Coefficient Interpretation Approach: A Bonus

We now have developed some intuition regarding why the estimated “slope” of the demand curve depends on the feed price reduced form (RF) coefficient estimates and why the estimated “slope” of the supply curve depends on the income reduced form (RF) coefficient estimates. The coefficient interpretation approach provides intuition and also gives us a bonus. The coefficient interpretation approach provides us with a simple way to derive the relationships between the estimated “slopes” of the demand and supply curves and the reduced form (RF) estimates. Compare the algebra we just used to express the estimated “slopes” of the demand and supply curves with the algebra used in Appendix 22.1.
Appendix 22.1 Algebraic Derivation of the Reduced From Equations

Demand Model: $Q_t^D = \beta_{Const}^D + \beta_P^D P_t + \beta_{Inc}^D + \epsilon_t^D$

Supply Model: $Q_t^S = \beta_{Const}^S + \beta_P^S P_t + \beta_{FeedP}^S + \epsilon_t^S$

Equilibrium: $Q_t^D = Q_t^S = Q_t$

Strategy to derive the reduced form (RF) equation for $P_t$:

- Substitute $Q_t$ for $Q_t^D$ and $Q_t^S$.
- Subtract the supply model equation from the demand model equation.
- Solve for $P_t$

\[
Q_t^D = \beta_{Const}^D + \beta_P^D P_t + \beta_{Inc}^D + \epsilon_t^D \\
Q_t^S = \beta_{Const}^S + \beta_P^S P_t + \beta_{FeedP}^S + \epsilon_t^S
\]

Substitute

\[
Q_t = \beta_{Const}^D + \beta_P^D P_t + \beta_{Inc}^D + \epsilon_t^D \\
Q_t = \beta_{Const}^S + \beta_P^S P_t + \beta_{FeedP}^S + \epsilon_t^S
\]

Subtract

\[
0 = \beta_{Const}^D - \beta_{Const}^S + \beta_P^D P_t - \beta_P^S P_t - \beta_{FeedP}^D + \beta_{FeedP}^S + \beta_{Inc}^D - \beta_{Inc}^S + \epsilon_t^D - \epsilon_t^S
\]

Solve

\[
\beta_P^S P_t - \beta_P^D P_t = \beta_{Const}^D - \beta_{Const}^S + \beta_P^D P_t - \beta_P^S P_t - \beta_{FeedP}^D + \beta_{FeedP}^S + \beta_{Inc}^D - \beta_{Inc}^S + \epsilon_t^D - \epsilon_t^S
\]

Strategy to derive the reduced form (RF) equation for $Q_t$:

- Substitute $Q_t$ for $Q_t^D$ and $Q_t^S$.
- Multiply the demand model equation by $\beta_P^S$ and the supply model equation by $\beta_P^D$.
- Subtract the new equation for the supply model from the new equation for the demand model.
- Solve for $Q_t$

\[
Q_t^D = \beta_{Const}^D + \beta_P^D P_t + \beta_{Inc}^D + \epsilon_t^D \\
Q_t^S = \beta_{Const}^S + \beta_P^S P_t + \beta_{FeedP}^S + \epsilon_t^S
\]
\[ Q_t = \beta^D_{\text{const}} + \beta^D_P P_t + \beta^s_P \text{Feed} P_t + \epsilon^D_t \]

Multiply

\[ \beta^s_p Q_t = \beta^s_P \beta^D_{\text{const}} + \beta^s_P \beta^D_P P_t + \beta^s_P \beta^s_P \text{Feed} P_t + \epsilon^D_t \]

Subtract

\[ \beta^s_p Q_t - \beta^D_p Q_t = \beta^s_P \beta^D_{\text{const}} - \beta^D_P \beta^s_{\text{const}} + \beta^s_P \beta^D_P P_t - \beta^s_P \beta^s_P P_t - \beta^D_P \beta^s_{\text{Feed} P_t} + \beta^s_P \beta^D_P \text{Inc} + \beta^s_P \epsilon^D - \beta^D_P \epsilon^s \]

Solve

\[ Q_t = \frac{\beta^s_P \beta^D_{\text{const}} - \beta^D_P \beta^s_{\text{const}}}{\beta^s_P - \beta^D_P} - \frac{\beta^D_P \beta^s_{\text{Feed} P_t}}{\beta^s_P - \beta^D_P} + \frac{\beta^s_P \beta^D_P \text{Inc}}{\beta^s_P - \beta^D_P} + \frac{\beta^s_P \epsilon^D - \beta^D_P \epsilon^s}{\beta^s_P - \beta^D_P} \]

\[ (\beta^s_P - \beta^D_P)Q_t = \beta^s_P \beta^D_{\text{const}} - \beta^D_P \beta^s_{\text{const}} + \beta^D_P \beta^s_P \text{Feed} P_t + \beta^s_P \beta^D_P \text{Inc} + \beta^s_P \epsilon^D - \beta^D_P \epsilon^s \]

1 The coefficients do not equal the slope of the demand curve, but rather the reciprocal of the slope. They are the ratio of run over rise instead of rise over run. This occurs as a consequence of the economist’s convention of placing quantity on the horizontal axis and price on the vertical axis. To avoid the awkwardness of using the expression “the reciprocal of the slope” repeatedly, we shall place the word slope within double quotes to indicate that it is the reciprocal.