Toric Varieties

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This errata sheet is organized by which printing of the book you have. The printing can be found by looking at the string of digits 10 9 8 ... at the bottom of the copyright page: the last digit that appears in this decreasing string indicates the printing.

Errata for the second printing as of May 3, 2021

Page 47, line 3 of Exercise 1.3.11: "$a+b \equiv 0 \bmod d$" should be "$a-b \equiv 0 \bmod d$"

Page 66, line −4: "Let u_F denote" should be "Let $u_F \in N$ denote"

Page 88, line 11: "$\sigma_{e_0} = \mathrm{Conv}(v_1,\dots,v_n)$" should be "$\sigma_{e_0} = \mathrm{Cone}(v_1,\dots,v_n)$"

Page 102, lines 13–14: "$X \times X \to \mathbb{C}$ whose fiber over 0" should be "$X \times X \to \mathbb{C}\times\mathbb{C}$ whose fiber over $(0,0)$"

Page 105, line 3 of part (a) of Exercise 3.0.9: (missing right parenthesis) "$\mathrm{Spec}(\mathbb{C}[x,y]/\langle y^2-x\rangle$" should be "$\mathrm{Spec}(\mathbb{C}[x,y]/\langle y^2-x\rangle)$"

Page 116, fourth bullet: "since $\sigma^\perp \subseteq \tau^\perp$" should be "since $U_\tau \subseteq U_\sigma$"

Page 121, first display: The union on the right should be $\displaystyle\bigcup_{\tau\preceq\sigma\preceq\sigma'} O(\sigma)$

Page 121, second display: At the end of the display, "$\mathsf{S}_{\sigma'}$" should be "$\mathbb{C}[\mathsf{S}_{\sigma'}]$"

Page 159, line 6 of the proof of Lemma 4.0.9: "dimension $< n$ should be "dimension $< \dim X$"

Page 176, Exercise 4.1.5: Delete the entire exercise and replace it with the following:

4.1.5. The weighted projective space $\mathbb{P}(q_0,\dots,q_n)$, $\gcd(q_0,\dots,q_n)=1$, is built from a fan in $N = \mathbb{Z}^{n+1}/\mathbb{Z}(q_0,\dots,q_n)$. Let $u_i \in N$ be the image of $e_i \in \mathbb{Z}^{n+1}$. The dual lattice is

$$M = \{(a_0,\dots,a_n) \in \mathbb{Z}^{n+1} \mid a_0q_0+\cdots+a_nq_n = 0\}.$$

Also assume that $\gcd(q_0,\dots,\widehat{q_i},\dots,q_n)=1$ for $i=0,\dots,n$.

(a) Prove that the u_i are the primitive ray generators of the fan giving $\mathbb{P}(q_0,\dots,q_n)$.

(b) Prove that $\mathrm{Cl}(\mathbb{P}(q_0,\dots,q_n)) \simeq \mathbb{Z}$. Hint: Show that the maps

$$M \longrightarrow \mathbb{Z}^{n+1} : m \longmapsto (\langle m,u_0\rangle,\dots,\langle m,u_n\rangle)$$
$$\mathbb{Z}^{n+1} \longrightarrow \mathbb{Z} : (a_0,\dots,a_n) \longmapsto a_0q_0+\cdots+a_nq_n$$

give an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Page 183, line 3: "$v \in M$" should be "$\mathrm{v} \in M$"

Page 188, Exercise 4.2.11: Delete the entire exercise and replace it with the following:

4.2.11. In Exercise 4.1.5, you showed that the weighted projective space $\mathbb{P}(q_0,\dots,q_n)$ has class group $\mathrm{Cl}(\mathbb{P}(q_0,\dots,q_n)) \simeq \mathbb{Z}$ when $\gcd(q_0,\dots,\widehat{q_i},\dots,q_n) = 1$ for all i. Prove that $\mathrm{Pic}(\mathbb{P}(q_0,\dots,q_n)) \subseteq \mathrm{Cl}(\mathbb{P}(q_0,\dots,q_n))$ maps to $m\mathbb{Z} \subseteq \mathbb{Z}$, where $m = \mathrm{lcm}(q_0,\dots,q_n)$. Hint: Show that $\sum_{i=0}^n b_i D_i$ generates the class group, where $\sum_{i=0}^n b_i q_i = 1$. Also note that $m \in M_{\mathbb{Q}}$ lies in M if and only if $\langle m, u_i\rangle \in \mathbb{Z}$ for all i, where the u_i are from Exercise 4.1.5.

Page 196, line 3: "elements of G" should be "elements of R^G"

Page 201, lines 5 and 6: "if its maximal connected solvable subgroup is a torus" should be "if the identity component of its maximal normal solvable subgroup is a torus".

Page 202, line 7 of the proof of Proposition 5.0.11: "closed in $\mathbb{C}^4$" should be "closed in X"

Page 213, Example 5.1.14: Delete the first paragraph of the example and replace it with the following:

Example 5.1.14. Fix positive integers $q_0,\dots,q_n$ with $\gcd(q_0,\dots,\widehat{q_i},\dots,q_n) = 1$ for all i and let $N = \mathbb{Z}^{n+1}/\mathbb{Z}(q_0,\dots,q_n)$. By Exercise 4.1.5, the images of the standard basis in $\mathbb{Z}^{n+1}$ give primitive elements $u_i \in N$ satisfying $q_0u_0 + \cdots + q_nu_n = 0$. Let Σ be the fan consisting of all cones generated by proper subsets of $\{u_0,\dots,u_n\}$.

Page 218, line 2 of Exercise 5.1.6: "$t = \prod_{i=0}^n t_i^{a_i}$" should be "$t = \prod_{i=0}^n t_i^{b_i}$"

Page 223, second display of Example 5.2.9, line 1: "$\pi(\mathbb{C}^2 \times \{0\})$" should be "$\pi(\{0\} \times \mathbb{C}^2)$"

Page 223, second display of Example 5.2.9, line 2: "$\pi(\{0\} \times \mathbb{C}^2)$" should be "$\pi(\mathbb{C}^2 \times \{0\})$"

Page 225, second line of Exercise 5.2.4: "blowup of $\mathbb{P}^2$" should be "blowup of $\mathbb{C}^2$"

Page 227, diagram in Exercise 5.3.2: Replace the diagram with

$$\begin{array}{ccc} (S_{x^{\hat\tau}})_0 & \longrightarrow & ((S_{x^{\hat\sigma}})_0)_{\pi_\sigma^*(\chi^m)} \\ \downarrow & & \downarrow \\ \mathbb{C}[\tau^\vee \cap M] & \longrightarrow & \mathbb{C}[\sigma^\vee \cap M]_{\chi^m}. \end{array}$$

Page 242, line after third display: "$\mathbb{Z} \oplus N/N_1$" should be "$\mathbb{Z} \oplus N_1/N$"

Page 244, line 1: "Use part (e)" should be "Use part (d)"

Page 246, line 2 of Example 6.0.2: "under inclusion" should be "under reverse inclusion"

Page 252, line 2 of part (b) of Proposition 6.0.16: "$s_i \in \mathscr{O}_X^r$" should be "$s_i \in \mathscr{O}_X(U_i)^r$"

Page 253, line 12 of the proof of Theorem 6.0.18: "$f_i f \in \mathscr{O}_X(D)(U_i)$" should be "$f_i f \in \mathscr{O}_X(U_i)$"

Page 253, line 16 of the proof of Theorem 6.0.18: "$s_i \in \mathscr{O}_X(D)(U_i)$" should be "$s_i \in \mathscr{O}_X(U_i)$"

Page 256, line 6 of the proof of Proposition 6.0.22: "$D' \neq Z_0$" should be "$D' \neq D_0$"

Page 260, line 1: “sections of $U_i \times \mathbb{C} \to \mathbb{C}$” should be “sections of $U_i \times \mathbb{C} \to U_i$”

Page 269, line -7: “$\phi_{D_P} = \phi_{\mathscr{O}_{X_P}(D_P),W} \longrightarrow \mathbb{P}^{s-1}$” should be “$\phi_{D_P} = \phi_{\mathscr{O}_{X_P}(D_P),W} : X_P \longrightarrow \mathbb{P}^{s-1}$”

Page 270, line 6: “$\mathrm{div}(s_i)_0$” should be “$\mathrm{div}_0(s_i)$”

Page 274, line -6: “be a Cartier divisor” should be “be an ample divisor”

Page 278, line -6: “Some of most interesting” should be “Some of the most interesting”

Page 280, line 3: “$m = m_\sigma$” should be “$m_\sigma = \mathsf{v}$

Page 284, line 7 of proof of Corollary 6.2.15: “$\langle v, u_\rho \rangle$” should be “$\langle \mathsf{v}, u_\rho \rangle$”

Page 285, line 3: The display should be “$D = D_3 + D_4 + 2D_5 + 2D_6$”

Page 286, part (a) of Exercise 6.2.8: It should be “$D = D_3 + D_4 + 2D_5 + 2D_6$”

Page 288, proof of Proposition 6.3.5: “C is smooth” should be “$\overline{C}$ is smooth”

Page 303, line 11: “$u_2 - 0 \cdot u_3 + u_4 = 0$” should be “$u_2 - 0 \cdot u_1 + u_4 = 0$”

Page 332, line -5: “full dimensional lattice polytope” should be “full dimensional lattice polyhedron”

Page 344, line 4 of Exercise 7.3.7: “normals scroll” should be “normal scrolls”

Page 351, line 14: “When we regard $\Omega_{S/\mathbb{C}}$ as an R-module” should be “Regarding S and $\Omega_{S/\mathbb{C}}$ as R-modules”

Page 356, Proposition 8.0.21: “*image*” should be “*the image*”

Page 358, line 2 of §8.1: “sheaves $\omega^1_{X_\Sigma}$” should be “sheaves $\Omega^1_{X_\Sigma}$”

Page 388, third to last display: Replace with

$$\begin{array}{ccccccc} \mathscr{F} & \longrightarrow & \mathscr{A}^0 & \xrightarrow{d^0} & \mathscr{A}^1 & \xrightarrow{d^1} & \cdots \\ \alpha \big\downarrow & & \alpha^0 \big\downarrow & & \alpha^1 \big\downarrow & & \\ \mathscr{G} & \longrightarrow & \mathscr{B}^0 & \xrightarrow{d^0} & \mathscr{B}^1 & \xrightarrow{d^1} & \cdots \end{array}$$

Page 389, line before Definition 9.0.1: “element of I” should be “element of $[\ell]$”

Page 399, five lines above Example 9.1.1: In two places, “$\mathscr{O}_X(D)$” should be “$\mathscr{O}_{X_\Sigma}(D)$”

Page 399, second display: In two places, “$\mathscr{O}_X(D)$” should be “$\mathscr{O}_{X_\Sigma}(D)$”

Page 426, line 6: “$H^p(X_\Sigma, X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell D - B))$” should be “$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell D - B))$”

Page 456, Exercise 9.5.8, line 10: “$H^1(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(a)) = 1$” should be “$H^1(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(a)) = 0$”

Page 470, first line of third display: “$P_s(b_s Q_s - Q_{s-1}) - (b_s P_s - P_{s-1}) Q_s$” should be “$P_s(b_{s+1} Q_s - Q_{s-1}) - (b_{s+1} P_s - P_{s-1}) Q_s$”

Page 471, line after first display: "$b_0 = b_1 = 2, b_2 = 3$" should be "$b_1 = b_2 = 2, b_3 = 3$"

Page 473, line 7: "$k\widetilde{k} \equiv 1 \bmod m$" should be "$k\widetilde{k} \equiv 1 \bmod d$"

Page 485, part (b) of Exercise 10.2.13: "in general" should be "when $\gcd(q_1, q_2) = 1$"

Page 485, line 1 of §10.3: "affine toric toric surfaces" should be "affine toric surfaces"

Page 509, line 11 of Example 10.5.7: "$D = 2D_1 + 2D_2$" should be "$D = 2D_1 + D_2$"

Page 521, line -1: "meet the relative interior of σ" should be "meet the relative interior of a cone $\tau \in \Sigma$ with $\sigma \preceq \tau$"

Page 522, lines 2–4: In four places, "σ" should be "τ"

Page 522, lines 4 and 5: Replace the final sentence of the proof with "Since $V(\sigma) = \bigcup_{\sigma \preceq \tau} O(\tau)$, the result follows easily (see Exercise 11.1.6)."

Page 565, four lines below the first display: "contained any" should be "contains any"

Page 683, display (14.1.4): "$S_{\ell d}$" should be "$R_{\ell d}$"

Page 683, two lines below display (14.1.4): "to global section" should be "to a global section"

Page 686, line 3 of proof of Lemma 14.2.1: "that Laurent monomial" should be "that the Laurent monomial"

Page 693, line 2 of Example 14.2.14: "$\mathbb{C}^{\Sigma}(1)$" should be "$\mathbb{C}^{\Sigma(1)}$"

Page 717, line -10: "Exercise 14.4.3" should be "in Exercise 14.4.3"

Page 769, third display, line 2: It should be "$\mathrm{Pic}(X_{\Sigma_+}) = \mathbb{Z}[D_1'],\ \ [D_1'] \mapsto (3, 0 \bmod 2)$"

Page 841, index entry for *weighted projective space*: Add pages 213, 225, 380, 384, and 485 to the list.

Errata for the first printing as of November 20, 2011

The errata in the first printing of the book include the errata for the second printing (listed above) together with the following additional errata, all of which were corrected in the second printing.

Page 70, line -1: Delete this line and replace it with the following:

We also have the following result about the Hilbert basis of $C(P) \cap (M \times \mathbb{Z})$.

Page 71, lines 1–7: Delete these lines and replace them with the following:

Lemma 2.2.16. Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be a lattice polytope of dimension $n \geq 2$ and let k_0 be the maximum height of an element of the Hilbert basis of $C(P)$. Then

$$k_0 \leq n - 1.$$

Proof. This follows easily from (2.2.3). □

The Hilbert basis of the simplex P of Example 2.2.15 has maximum height 2 by Lemma 2.2.16. The paper [**187**] gives a version of Lemma 2.2.16 that applies to Hilbert bases of more general cones. See also Exercise 2.2.9.

Page 74, Exercise 2.2.9: Delete the entire exercise and replace it with the following:

2.2.9. Let $\mathrm{Cone}(\mathscr{A}) \subseteq M_{\mathbb{R}} \simeq R^n$ be a full dimensional strongly convex cone and let $\mathscr{H}$ be the Hilbert basis of $\mathrm{Cone}(\mathscr{A}) \cap M$. Write $\mathscr{A} = \{m_1, \dots, m_s\} \subset M$, where each m_i is nonzero and primitive. Following [**187**], the *height* of $m \in \mathrm{Cone}(\mathscr{A}) \cap M$ is defined to be

$$h(m) = \max\big(\textstyle\sum_{j=1}^n \lambda_j \mid m = \sum_{j=1}^n \lambda_j m_{i_j},\ m_{i_1}, \dots, m_{i_n} \text{ linearly independent}\big).$$

The main result of [**187**] states that if $n \geq 3$, then $h(m) < n-1$ for all $m \in \mathscr{H}$.
(a) Let $C(P) \subseteq M_{\mathbb{R}} \times \mathbb{R}$ be the cone from Lemma 2.2.16. Prove that $h((m,k)) = k$ for all $(m,k) \in \mathrm{Cone}(P) \cap M \times \mathbb{Z}$.
(b) Explain how Lemma 2.2.16 follows from the result of [**187**].

Page 113, line -3: "(*that is, X_Σ has*" should be "(*i.e., X_Σ has*"

Page 113, line -2: Add the following new sentence at the end of the existing line: "*We say that X_Σ is **simplicial** in this case.*

Page 115, line -4: "the *relative interior* of" should be "the relative interior of"

Page 136, line 2 of Exercise 3.3.5: "with the property that" should be "such that $\overline{\phi}(\widehat{\sigma} \cap N) = \sigma' \cap N'$ and"

Page 137, line 3 of part (b) of Exercise 3.3.7: "$\overline{N''}$" should be "$\overline{\phi}^{-1}(N'')$"

Page 138, line 1 of Exercise 3.3.12: "Consider the fan" should be "Consider the complete fan"

Page 184, line -10: "$-\sum_\rho \varphi_D(u_\rho) D_\rho$" should be "$-\sum_\rho \varphi(u_\rho) D_\rho$"

Page 187, part (a) of Exercise 4.2.7: "$O(\sigma) =$" should be "$V(\sigma) = \overline{O(\sigma)} =$"

Page 188, line 2 of Exercise 4.2.9: "instead use" should be "use"

Page 188, line 3 of Exercise 4.2.9: "with $a > 1$" should be "with a odd"

Page 224, line -1: "where $\sigma_1, \sigma_2 \in \Sigma$ are as in Example 5.1.16" should be "where $\sigma_i = \mathrm{Cone}(u_0, u_i)$ for u_0, u_1, u_2 as in Example 5.1.16"

Page 226, line 2 of Exercise 5.2.6: Add the new sentence: "Let σ_1, σ_2 be as in Example 5.2.11."

Page 228, line 2 of proof of Lemma 5.3.5: "for all ρ" should be "for all $\rho \in \sigma(1)$"

Page 249, third display: "$\mathrm{Hom}_{\mathscr{O}_X(U)}(\mathscr{F}(U), \mathscr{G}(U))$" should be "$\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U)$"

Page 276, line 1 of Exercise 6.1.2: "(b) $\Rightarrow$ (d)" should be "(b) $\Rightarrow$ (f)"

Page 309, Exercise 6.4.2: In two places, "Example 6.4.2" should be "Example 6.4.6"

Page 331, line -1: "$k \geq n$" should be "$k \geq n-1$"

Page 343, line 1: "rays in P" should be "rays in Q"

Page 379, lines 2 and 3 of Exercise 8.2.14: Replace the sentence beginning "In the discussion" with "Let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope containing the origin as an interior point. In the discussion of the algebra (8.2.5), we saw that P gives the cone"

Page 382, part (a) of Lemma 8.3.6: "common edge" should be "common facet"

Page 384, line -14: "most more recent" should be "most recent"

Page 385, line 4: "contains exactly two vertices" should be "contains exactly two lattice points"

Page 398, line 3 of §9.1: "torus-invariant $\mathbb{Q}$-Cartier divisors" should be "torus-invariant Weil divisors"

Page 398, line -13: "write these as σ_i and" should be "write these as $\sigma_1,\ldots,\sigma_\ell$ and"

Page 398, line -11: "torus-invariant Cartier divisor" should be "torus-invariant Weil divisor"

Page 410, line 10: "$\langle m,v\rangle < \varphi_D(u)$" should be "$\langle m,v\rangle < \varphi_D(v)$"

Page 420, line 1 of part (b) of Exercise 9.2.12: "σ is contained" should be "Δ_σ is contained"

Page 450, line -7: "$\mathrm{Ext}_S^{p+1}(S/B(\Sigma)^{[k]},S)$" should be "$\mathrm{Ext}_S^{p+1}(S/B(\Sigma)^{[k]},S)_\alpha$"

Page 477, part (c) of Theorem 10.2.12: "*(resp.* Θ_0*)*" and "*(resp. even)*" should be "(*resp.* Θ_0)" and "(*resp. even*)"

Page 496, line -10: "$b_1 > 2$" should be "$b_1 \geq 2$"

Page 496, line -9: "$b_1 < -2$" should be "$b_1 \leq -2$"

Page 501, line 2 of Exercise 10.4.2: "$b_1 < 2$" should be "$b_1 \leq 2$"

Page 529, line -5: "dimension ≥ 3" should be "dimension ≥ 4"

Page 705, fourth bullet of Example 14.3.8: "$(\mathbb{C}^3)^{\mathsf{ss}}$" and "$(\mathbb{C}^3)^{\mathsf{s}}$" should be "$(\mathbb{C}^3)^{\mathsf{ss}}_\chi$" and "$(\mathbb{C}^3)^{\mathsf{s}}_\chi$"

Page 797, bottom: Add the following new paragraph:

New packages and updates to old packages (including new URLs) can be found at `http://www.cs.amherst.edu/~dac/toric.html`.

Page 817, reference [2]: "K. Matsuk" should be "K. Matsuki"

Page 819, reference [40]: "vectors bundles" should be "vector bundles"

Page 819, reference [56]: "W. Brun" should be "W. Bruns"

Page 839, index entry for *simplicial toric variety*: "180" should be "113, 180"