

MA904 - Functions and Equations for Students of Calculus- June 25-29, 2007

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Overview

Meeting Times: M-F 12:30-3:30pm

Location: Barus Holley room 163

Instructor: Ben Hutz

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Textbook: None.

This is a one week course for students who plan to take calculus next year or the year after. The only prerequisites are basic algebra and a willingness to work. One of our two main topics will be functions. We will start with the basic abstract definition and work towards complex examples and special functions such as logarithms. An understanding of these functions will allow you to focus on the concepts when they appear in calculus courses. Our other main topic is equations. We will mainly be interested in writing equations to describe situations and solving them for certain quantities. This is a major area of difficulty for many calculus students.

Course Goals

- (1) Understand what a function is and be able to describe situations through functions.
- (2) Understand the relationship and differences between logarithms and exponential functions.
- (3) Be able to describe situations through equations.
- (4) Be able to solve complicated equations for specific quantities.

Expectations

Mathematics is a very personal discipline that is best learned by doing rather than by observing. Consequently much of this week will be spent doing problems. I will do some as examples, you will do some during class exercises, and there will be homework problems due on Tuesday, Wednesday, and Thursday. A major component of this class will be a project that you will work on throughout the week and present your results on Friday. See more about the projects below.

Class Attendance

Class attendance is mandatory for you to receive a certificate of completion.

Homework

You will be writing complete solutions to two of the group problems worked on in class. Which problems you may choose from will be stated in class each day. The purpose of this is to help you learn to write good correct solutions to problems. I will be grading these for how the solution is written in addition to correctness of the final answer.

Here are a few guidelines for the presentation of your solutions.

1. Your writing must be clear and legible.
2. Your homework should be well-written, using complete sentences to justify your results where necessary.
3. Write your solutions so that you could hand them to another student in the class and he/she could understand your explanation.
4. If you write in pen, there should be no scratch-outs.
5. Do not turn in paper torn from a spiral notebook with ragged edges. Scissors are a technology that you can use to solve this problem.
6. Clearly label each problem and solution.

Projects

There will be three different projects that you can choose from and as many people as want to can work on each project. You are encouraged to work together and to talk with me often about your progress and for guidance. On Friday you will present your results.

Getting Help

Feel free to contact me via e-mail during the week, especially about your projects. I will also be in my office for additional help at times to be announced in class.

1 Monday

1.1 Definition of a function

By the end of today you should be comfortable with the concepts of a function, domain and range of a function, 1-1 and inverse functions, and polynomial/rational functions.

Definition. A *function* is a rule which assigns an output to a given input. The possible input values are called the *domain* of the function and the possible output values are called the *range* of the function.

Example 1.1. Define a function f that assigns to each letter of the alphabet its place value. In other words,

$$\begin{aligned}f(A) &= 1 \\f(B) &= 2 \\&\vdots \\f(Z) &= 26.\end{aligned}$$

The domain of this function is

$$\{A, B, C, D, E, \dots, Z\}$$

and the range of this function is

$$\{1, 2, 3, 4, \dots, 26\}.$$

Example 1.2. We could also define a function that takes shapes to colors. Such as

$$\begin{aligned}f(\text{square}) &= \text{red} \\f(\text{rectangle}) &= \text{blue} \\f(\text{circle}) &= \text{green} \\&\vdots\end{aligned}$$

The domain of this function is shapes and the range of this function is colors.

Example 1.3. A function that looks more familiar may be one which takes numbers to numbers, such as by the rule

$$f(x) = x^2.$$

Meaning that the function f takes any number to its square.

$$\begin{aligned}f(2) &= 4 \\f(1.5) &= 2.25 \\&\vdots\end{aligned}$$

The domain and range of this function are all numbers.

While the first two example functions may seem arbitrary, and they are to some extent, they are still functions. The third function is perhaps what you are used to seeing, and is in fact the kind of function we will be most concerned with in this course.

1.2 Algebra of Functions

Functions can be added, subtracted, multiplied, and divided (if their domain and ranges match appropriately). They can also be composed. We need to define all of these operations.

Assume that f and g are functions. Then we can define arithmetic of f and g as follows:

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \\(f - g)(x) &= f(x) - g(x) \\(f \cdot g)(x) &= f(x) \cdot g(x) \\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \\(f \circ g)(x) &= f(g(x)).\end{aligned}$$

So that these abstract definitions make sense, let's consider the concrete example

$$f(x) = x^2 \quad \text{and} \quad g(x) = x + 1.$$

The we have

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) = x^2 + x + 1 \\(f - g)(x) &= f(x) - g(x) = x^2 - x - 1 \\(f \cdot g)(x) &= f(x) \cdot g(x) = x^3 + x^2 \\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} = \frac{x^2}{x + 1} \\(f \circ g)(x) &= f(g(x)) = f(x + 1) = (x + 1)^2 = x^2 + 2x + 1.\end{aligned}$$

Notice that

$$x^2 + 2x + 1 = (f \circ g)(x) \neq (g \circ f)(x) = x^2 + 1.$$

In other words, the composition of the functions usually (but not always) depends on the order of the composition.

1.3 1-1 and Inverse Functions

A function is *1-1* (one-to-one) if every output value corresponds to exactly one input value. If you think of the function as a graph $y = f(x)$ this means that if you draw a horizontal line it will intersect the graph in at most one point.

An *inverse function* is a function that takes the output values back to the input values. So in particular, if f and g are inverse functions, then

$$(f \circ g)(x) = f(g(x)) = x$$

and

$$(g \circ f)(x) = g(f(x)) = x.$$

For example the function

$$f(x) = x + 1$$

is 1-1 and has the inverse function

$$g(x) = x - 1.$$

For a function to have an inverse function, it must be one-to-one. Otherwise, there is more than one input value associated to every output and no way to choose.

For example, the function

$$f(x) = x^2$$

is not 1-1 since

$$f(-1) = 1 = f(1).$$

Assume that there exists an inverse function $g(x)$ for $f(x)$. Then what is the value of $g(1)$? If we choose $g(1) = 1$ then we have

$$(g \circ f)(-1) = 1$$

which contradicts the fact that g is an inverse function. Hence there is no inverse function for x^2 .

The problem with $f(x) = x^2$ was the fact that a horizontal line may intersect the graph in more than one point (i.e., f is not one-to-one). However, we can work around this problem by restricting the domain of f to a set of values where it is 1-1. For example, if we restrict to positive values of x only, then $f(x)$ is 1-1 and its inverse function is $g(x) = \sqrt{x}$.

In general, the way to find an inverse function is to consider the equation

$$y = f(x)$$

and solve for $x = g(y)$.

Example 1.4. Consider the function $f(x) = \frac{x^2+1}{3}$. Then we have

$$y = \frac{x^2 + 1}{3}$$

$$3y = x^2 + 1$$

$$3y - 1 = x^2$$

$$\sqrt{3y - 1} = x.$$

Hence we have the inverse function

$$g(x) = \sqrt{3x - 1}.$$

1.4 Two Special Classes of Functions

1.4.1 Polynomial Functions

A polynomial is a variable x is a combination of constants and positive powers of x . For example

$$\begin{aligned}x^2 + 1, \\x + 2, \\x, \\x^3 + x + 1, \\x^{50} - 2x^5 + 2\end{aligned}$$

are all polynomials, but

$$\begin{aligned}x^{1/3} + 1, \\ \frac{1}{x}, \\ \sin x\end{aligned}$$

are not.

A polynomial function is a function that may be expressed as a polynomial. Note that a polynomial function may not always look like a polynomial until you simplify it. For example

$$\begin{aligned}\frac{x^3 - 5x^2 + 7x - 12}{x - 4} = x^2 - x + 3 \quad \text{and} \\ \frac{x^{3/2}}{x^{1/2}} = x\end{aligned}$$

are both polynomial functions.

1.4.2 Rational Functions

A rational function is the ratio of two polynomial functions. For example

$$\frac{x^2 + 1}{x + 1} \quad \text{and} \quad \frac{x^3 - 1}{x^7 + 3x^3 + 7x + 1}$$

are both rational functions.

1.5 Problems I

Problem 1.1. Simplify the following.

1. x^3x^4
2. $(x^3)^4$
3. 4^{-2}
4. $\frac{4x^2y^{-1}}{2x^2y^{-2}}$
5. $(5a - 3b)^2$
6. $7x^3(4x^2 - 5x) - 2x^2(3x^3 - 6x^2)$
7. $(2x + y)^2 - (2x - y)^2$
8. $8^{2/3}$
9. $\sqrt{\frac{36x^3}{9x}}$
10. $(4x^3y^8)^3$
11. $\left(\frac{3x^{-1}y^2}{2x^{-2}y^0}\right)^{-4}$

Problem 1.2. Find the domain and range of the following functions.

1. $f(x) = 2x + 1$
2. $f(x) = \sqrt{x^2 - 4}$
3. $f(x) = \sqrt{\frac{x-2}{x-1}}$
4. $f(x) = \frac{x^2-1}{x-1}$

Problem 1.3. Perform the following operations for $f(x) = x + 2$ and $g(x) = 2x^2$.

1. $f(2) - g(3)$
2. $g(x^2)$
3. $\frac{f(1)}{g(2)}$

Problem 1.4. Compute $f \circ g$ and $g \circ f$ for the following pairs of functions.

- (a) $f(x) = x^2 + 1, g(x) = \sqrt{x}$
- (b) $f(x) = x - 1, g(x) = x^2$

Problem 1.5. Find two functions whose composition is the given function.

- (a) $\sqrt{(x+1)^3}$

(b) $x^3 - 1$

(c) $(x + 1)^2 + 2$

Problem 1.6. Find inverse functions (if possible) to the following functions.

1. $f(x) = x^3 + 1$

2. $f(x) = \frac{2x-3}{4}$

3. $f(x) = \sqrt[3]{x+2} - 2$

1.6 Problems II

Problem 1.7. Suppose $f(x, y)$ is defined by

$$f(x, y) = x + y + xy$$

Given $f(3, y) = 23$, find the value of y .

Problem 1.8. Suppose that f is function such that

$$f(x + y) = f(x) + f(y) + 1$$

for all x and y and also $f(1) = 2$. Then what is $f(3)$?

Problem 1.9. Define a function

$$f(x, y) = \frac{x + y}{x - y}$$

What is the value of $f(f(1, 2), 3)$?

Problem 1.10.

$$f(x, y) = x^3 - y$$

What is $f(a, f(a, a))$?

Problem 1.11. Define

$$f(x, y) = (x + y)(x - y)$$

for all x, y . What is $f(3, f(4, 5))$?

Problem 1.12. Define

$$f(x, y) = \frac{1 + x}{1 + y^2}$$

what is the value of $f(f(2, 0), 1)$?

Problem 1.13. If $f(1) = 2$ and $f(n) = f(n - 1) + \frac{1}{2}$ what is $f(101)$?

1.7 Problems III

Problem 1.14. Define

$$f(x) = 1 - x \quad \text{and} \quad g(x) = \frac{1}{x}.$$

You can compose these functions in two ways

$$f(g(x)) \quad \text{and} \quad g(f(x)).$$

You can compose each of these two new functions with themselves, and also the original functions in a number of ways. Keep composing these functions with new ones as they are generated and figure out simplified formulae for them in terms of x . (Don't forget to include compositions such as $f(f(x))$.)

1. How many distinct functions are there, including f and g themselves?
2. List them.
3. How is each one composed from f and g .
4. How do you know that these are all of them?
5. For what real numbers are all of these functions simultaneously defined?

Problem 1.15. Define

$$f(x) = \frac{x+1}{-x+1} \quad \text{and} \quad g(x) = -x.$$

You can compose these functions in two ways

$$f(g(x)) \quad \text{and} \quad g(f(x)).$$

You can compose each of these two new functions with themselves, and also the original functions in a number of ways. Keep composing these functions with new ones as they are generated and figure out simplified formulae for them in terms of x . (Don't forget to include compositions such as $f(f(x))$.)

1. How many distinct functions are there, including f and g themselves?
2. List them.
3. How is each one composed from f and g .
4. How do you know that these are all of them?
5. For what real numbers are all of these functions simultaneously defined?

Problem 1.16. The function $f(x, y)$ satisfies

1. $f(0, y) = y + 1$

2. $f(x + 1, 0) = f(x, 1)$

3. $f(x + 1, y + 1) = f(x, f(x + 1, y))$

for all non-negative integers x and y .

(a) Determine $f(4, 0)$.

(b) Determine $f(1, y)$.

(c) Determine $f(2, y)$.

2 Tuesday

Today we will look closely at the case of polynomial functions and radical expressions.

2.1 Roots of Polynomials

We will denote a general polynomial as

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

where at least c_n is not zero. Then we say that $f(x)$ has degree n and has coefficients c_0, \dots, c_n . A value a is root of $f(x)$ if

$$f(a) = 0.$$

Notice, that every value for which the graph $y = f(x)$ intersects the x -axis is a root of $f(x)$.

Example 2.1. For example

$$f(x) = x^2 - 1$$

is degree 2 with coefficients $\{1, 0, -1\}$. It has two roots, $\{-1, 1\}$.

We are interested in roots, because if we can describe a problem with a polynomial equation, finding the roots of the polynomial is the same as solving the equation.

Example 2.2. For example solving the equation

$$x^2 - 2x - 6 = x + 7$$

is the same as finding the roots of

$$x^2 - 3x - 13$$

One question you may ask yourself, is how many (if any) roots of a polynomial are there?

The answer to this question comes in the form of the fundamental theorem of algebra, which states that for any polynomial of degree n there are n complex numbers, a_1, \dots, a_n such that

$$f(x) = (x - a_1) \cdots (x - a_n).$$

Note that this implies that the a_i are all of the roots of the polynomial $f(x)$ and $f(x)$ is divisible by $(x - a_i)$ if and only if a_i is a root of $f(x)$. So the problem of finding roots of a polynomial is the same problem as factoring the polynomial.

Example 2.3. The polynomial $f(x) = x^2 - 1 = (x-1)(x+1)$ has roots $\{1, -1\}$.

Recall that the quadratic equation gives the roots of a degree 2 polynomial in terms of its coefficients. For $f(x) = ax^2 + bx + c$ we have

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for its roots.

2.2 Radical Expressions

We first recall some rules and common mistakes when working with radicals. The notation

$$\sqrt[n]{}$$

means to take the n th root of whatever is under the radical. So for example

$$y = \sqrt[5]{x}$$

implies that

$$x = y^5.$$

Here are some common operations with radicals.

1. $\sqrt[n]{x} \sqrt[n]{y} = \sqrt[n]{xy}$
2. $y \sqrt[n]{x} = \sqrt[n]{y^n x}$
3. $(\sqrt[n]{x})^n = x$
4. $\sqrt[n]{x} = x^{\frac{1}{n}}$.

We can use these to solve equations as in the following example.

Example 2.4. Solve for x :

$$\sqrt{x+2} = x.$$

Squaring both sides we have

$$x+2 = x^2$$

which is the same as

$$x^2 - x - 2 = 0$$

which has solutions

$$x = \{2, -1\}.$$

2.3 Problems I

Problem 2.1. Solve $LI^2 + RI + 1/C = 0$ for I

Problem 2.2. Solve $(1/p + 1/q) = 1/f$ for q

Problem 2.3. Solve the following equation for y .

$$\sqrt[4]{y^2 + 2} = \sqrt{x + \sqrt{x + 1}}.$$

Problem 2.4. List all possible rational zeros of $4x^3 + 5x^2 + 7x - 18$ (you do not need to actually perform the trial division).

Problem 2.5. Determine by trial division if the following polynomials have any real roots.

1. $x^5 - x^4 + 2x^3 - 2x^2 + x - 1$
2. $3x^{12} + 18x^{10} + 45x^8 + 60x^6 + 45x^4 + 18x^2 + 3$

Problem 2.6. Simplify the following

- (a) $\frac{x^{p+q}}{x^{p-q}}$
- (b) $\sqrt[n]{\frac{x^{mn}}{x^{n^2}}}$
- (c) $(x^{p+1})^2(x^{p-1})^2$

Problem 2.7. Simplify

$$\sqrt{\frac{x}{1 - \frac{x-1}{x}}}$$

Problem 2.8. Find the quotient and remainder when $3x^5 - 7x^3 + 5x^2 + 6x - 6$ is divided by $x^3 - x + 2$.

Problem 2.9. Factor the following.

1. $x^2 - 7x - 18$
2. $3x^2 - 5xy - 12y^2$
3. $8x^3 - 27$
4. $x^6 + 1$

Problem 2.10. Simplify

$$\frac{2x^4 - 4x + 9}{x^2 + 2x - 1}$$

Problem 2.11. For what real value(s) of x is

$$\sqrt{x + \sqrt{2x - 1}} + \sqrt{x - \sqrt{2x - 1}} = 2,$$

2.4 Problems II

Problem 2.12. (a) We know a quadratic polynomial has two roots. Write its coefficients in terms of its roots. (i.e. expand $(x - a_1)(x - a_2)$).

(b) Do the same for a cubic polynomial?

(c) Can you guess the general formula?

Problem 2.13. The equation

$$\frac{2003}{2004}x + 1 + \frac{1}{x}$$

has two roots. Label them a, b . What is $\frac{a+b}{ab}$?

Problem 2.14. Let a, b denote the solutions of $2x^2 + 3x - 5 = 0$. What is $(a - 1)(b - 1)$?

Problem 2.15. Let $f(x) = x^2 + x$. Show that $4f(a) = f(b)$ has no solutions in positive integers a and b .

Problem 2.16. Simplify

$$\sqrt[3]{x \sqrt{x \sqrt{x \sqrt{x \sqrt{x}}}}}$$

Problem 2.17. Let a and b be the roots of the equation $x^2 - mx + 2$. Suppose that $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of the equation $x^2 - px + q$. What is q ?

Problem 2.18. There are two values of a such that

$$4x^2 + ax + 8x + 9 = 0$$

has only one solutions. What are those two values?

2.5 Problems III

Problem 2.19. If $x^3 + px^2 + qx + r = 0$ has roots x_1, x_2, x_3 calculate

$$(x_1x_2)^2 + (x_1x_3)^2 + (x_2x_3)^2.$$

Problem 2.20. (calculator) Estimating the zero of a polynomial:

Consider the polynomial $f(x) = x^3 + 3x - 1$. Your goal is to approximate a zero of this function to within 10^{-2} .

1. To begin show that $f(a) = 0$ for some a in the interval $[-1, 1]$.
2. Now bisect the interval $[-1, 1]$ and determine which half contains a zero of $f(x)$.
3. How many times must you repeat this bisection process to have a sufficiently accurate (within 10^{-2}) answer.
4. Find such an answer.

Problem 2.21. Let x_1, x_2 be the roots of $x^2 + ax + 1$ and x_3, x_4 be the roots of $x^2 + bx + 1$. What is

$$\frac{x_1}{x_2x_3x_4} + \frac{x_2}{x_1x_3x_4} + \frac{x_3}{x_1x_2x_4} + \frac{x_4}{x_1x_2x_3}$$

Problem 2.22. Show that by adding 1 to the product of 4 consecutive integers a perfect square is obtained.

Problem 2.23. Find all numbers x such that

$$\sqrt{5 - \sqrt{5 - x}} = x.$$

3 Wednesday

Today we will define exponential and logarithmic functions and discuss problems that concern exponential growth and decay, such as population modelling.

3.1 Exponential and Logarithmic Functions

Exponential functions are functions of the form

$$f(x) = a^x$$

for some constant a called the *base*.

Exponential functions have inverse functions called logarithmic functions.

$$g(x) = \log_a(x)$$

where again a is a constant called the *base*. We have by the definition of inverse functions that $\log_a(a^x) = x$. In other words, if

$$\log_a(x) = y,$$

then

$$a^y = x.$$

Rules of Logarithms

- (a) $\log xy = \log x + \log y$
- (b) $\log x^a = a \log x$
- (c) $\frac{\log_a x}{\log_b a} = \log_b x$.

The most common base used is $e \sim 2.718281828459\dots$, which has as inverse $\log_e(x) = \ln(x)$ called the *natural log*. There are several reason why this function is given special consideration. Perhaps the most relevant to us is because it solves the following equation:

$$\text{rate of change of } f(x) = kf(x)$$

for some constant k . The solution is

$$f(x) = Ce^{kx}$$

where $C = f(0)$ is the initial value. This type of equation is what is known as exponential growth (or decay) and k is the constant of growth (or decay).

Example 3.1. If a population has an initial population of 100 and has an exponential growth rate of 2, how many people are there after 5 years?

$$y = 100e^{2t}$$
$$y(5) = 100e^{10}$$

Example 3.2. If a population has 200 people in year 1 and 1600 people in year 4 how many people did it have in year 3?

$$Ce^{k1} = 200, Ce^{k4} = 1600$$

$$8 = e^{3k}$$

$$k = (\ln 8)/3 = \ln 2$$

$$C = 100$$

$$100e^{3 \ln 2} = 800$$

3.2 Problems I

Problem 3.1. Solve the following equations.

1. $3^x = 243$
2. $2^{3x-1} = 2^x$
3. $\frac{4^x}{4^{2x}} = 64$
4. $\left(\frac{1}{2}\right)^x = 32$

Problem 3.2. Write the following as a single logarithm.

1. $2 \log_3 x + \log_3 y$.
2. $2 \log x - 3(\log y - \log z)$
3. $3(\log_2 x + \log_2 y - \log_2 z)$.

Problem 3.3. Solve the following equations.

1. $\log_5(x + 2) = 1$.
2. $\log_3 x = \log_3 2 + \log_3(x^2 - 3)$.
3. $\ln 5 - \ln x = -1$.
4. $3 \ln 2 + \ln(x - 1) = \ln 24$.

Problem 3.4. Define

$$f(x) = 4^x.$$

Then what is $f(x + 1) - f(x)$?

Problem 3.5. A bacteria culture grows exponentially so that the initial number has doubled in 3 hours. How many times the initial number will be present after 9 hours?

Problem 3.6. If the world population in 1980 was 4.5 billion and if it is growing exponentially with a growth constant $k = 0.04 \ln 2$, find the population in the year 2030.

Problem 3.7. An amount of money y_0 earning 5% per year is compounded continuously. How much is available after 20 years?

Problem 3.8. If a quantity of money, earning interest compounded continuously, has grown 55 times the initial amount after 100 years, what was the yearly interested rate?

Problem 3.9. A certain chemical decomposes exponentially. Assume that 200 grams becomes 50 grams in 1 hour. How much will remain after 3 hours.

Problem 3.10. Assume that a quantity y of a certain substance decays exponentially, with decay constant k . Let y_0 be the initial quantity at time 0. At what time T will only half of the original quantity remain.

Problem 3.11. Find all solutions to

(a) $e^{2x} - 3e^x + 2 = 0$

(b) $\ln(x + 1) + \ln x = \ln 2$

Problem 3.12. Solve the following equations for x and also for y .

(a) $\sqrt[3]{y} = \sqrt{x + 1}$

(b) $\sqrt[3]{y + 1} = a^x$

(c) $\sqrt{y^2 + 2y + 1} = \ln x + \ln(x + 1) + 1$

3.3 Problems II

Problem 3.13. If the population of a certain island has 60 in year 1, 80 in year 2 and 120 in year 3, is the population growing exponentially?

Problem 3.14. If the population of Latin America has a doubling time of 27 years, by what percent does it grow per year?

Problem 3.15. Consider the following equation

$$P(t) = \frac{C}{1 + \left(\frac{C}{P_0} - 1\right)e^{-kt}}.$$

This equation also models population with P_0 representing the initial population and t representing time. Describe what happens to the population over time in the following cases.

1. $C > P_0$
2. $C = P_0$
3. $C < P_0$

What does the constant C represent?

Problem 3.16. Functions that convert addition to multiplication:

Suppose that a certain city has an initial population of 10,000 and that the population increases by 10% each year. Let $P(x)$ be the population of this city after x years.

1. What will the population be after 3 years?
2. Check that

$$P(2) = \frac{1}{10,000}P(1)P(1)$$

and that

$$P(3) = \frac{1}{10,000}P(2)P(1)$$

3. Show that in general we have

$$P(x+y) = \frac{1}{10,000}P(x)P(y).$$

4. Show that any exponential function $f(x) = a^x$ satisfies this property.

Problem 3.17. (calculator) Can you find any value (x, y) with $x \neq y$ such that

$$x^y = y^x?$$

4 Thursday

Today we will talk about systems of equations and how to solve them using elimination.

4.1 Systems of Equations

A *system of equations* is a set of equations that you are trying to solve simultaneously.

Example 4.1. For example,

$$\begin{aligned}x + y &= 1 \\x - y &= 2\end{aligned}$$

is a system with two equations and two unknowns. The equations are $x + y = 1$ and $x - y = 2$. The unknowns are x and y . The unique solution to this system of equations is $(x, y) = (1, -1)$.

Graphically, a solution to a system of equations represents the intersection points of the graphs of the equations. In the above example, we have two linear equations, whose graphs are therefore lines and it is clear that lines intersect in zero, one, or infinitely many points. These particular lines happen to intersect at the single point $(1, -1)$.

4.2 Linear Systems of Equations

Perhaps the simplest system of equations is called a *linear system of equations*. As you may guess, this means that every equation in the system is a linear polynomial in the unknowns.

For a system to have a unique solution you must have at least as many equations as variables. The method we will use for solving systems of equations is called elimination. In other words, we use the equations to successively solve for the different variables.

Example 4.2. Consider the equations

$$\begin{aligned}3x + 2y &= 9 \\2x + y &= 12\end{aligned}$$

We can solve the second equation for y to get

$$y = 12 - 2x$$

and substitute this into the first equation to get

$$3x + 24 - 6x = 9.$$

Solving this equation for x we get

$$x = 5.$$

Returning to the relation

$$y = 12 - 2x$$

we get

$$y = 2.$$

Hence the solution is $(x, y) = (5, 2)$.

4.2.1 Complex Numbers

One way to deal with complex numbers is to treat i like a variable x which satisfies $i^2 = -1$ and treat a complex number $z = a + bi$ as a linear polynomial. Then problems come down to solving a system of linear equations for the coefficients.

Let $i = \sqrt{-1}$. Then a *complex number* has the form $z = a + bi$. We can add (1) and multiply (2) complex numbers.

$$(1) \quad a + bi + c + di = a + b + (c + d)i$$

$$(2) \quad (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Example 4.3. Given $1 + 2i$ find z such that $(1 + 2i)z = 2$. Assuming that $z = a + bi$ we need to solve for a and b . Multiplying the left hand side we get

$$(1 + 2i)(a + bi) = a - 2b + (b + 2a)i.$$

Setting this expression equal to 2 means that we must have

$$a - 2b = 2$$

$$2a + b = 0$$

which is a system of linear equations which we can solve to get $a = \frac{2}{5}$ and $b = -\frac{4}{5}$. So we have the solution $z = \frac{2}{5} - \frac{4}{5}i$.

4.3 Higher order systems of equations

We can increase the complexity by considering not just linear equations, but systems of more and more equations in more and more unknowns with higher and higher degrees. However, the method remains the same. Successively eliminate the variables until you are down to a single equation in a single unknown.

Example 4.4. Consider the system

$$\begin{aligned}x^3 + y &= 1 \\y^2 + 2x^3 &= 2.\end{aligned}$$

We can solve the first equation for y to get

$$y = 1 - x^3.$$

Substituting this into the second equation yields

$$(1 - x^3)^2 + 2x^3 = 2.$$

Simplifying we get

$$1 - 2x^3 + x^6 + 2x^3 = 2.$$

And finally

$$x^6 = 1$$

Which has the solutions $\{1, -1\}$. (The other 4 solutions are complex numbers.)

All of the systems of equations we have seen so far have involved polynomials. It is perfectly valid to have a system of equations with any equations. However, there may or may not be a solution.

The systems of equations

$$\begin{aligned}\sin x + \cos y &= 1 \\ x + e^y &= 2\end{aligned}$$

and

$$\begin{aligned}\ln x &= e^y \\ \sin(\cos(x) + y) &= \sqrt{\tan x}\end{aligned}$$

are difficult to solve explicitly.

4.4 Problems I

Problem 4.1. Solve.

$$\begin{aligned}x + y &= 1 \\x - y &= 3\end{aligned}$$

Problem 4.2. Solve

$$\begin{aligned}x + y &= 1 \\2x + 2y &= 2\end{aligned}$$

Problem 4.3. Solve

$$\begin{aligned}x + y &= 1 \\2x + 2y &= 3\end{aligned}$$

Problem 4.4. Solve

$$\begin{aligned}x + y + z &= 2 \\2x + 3y + z &= 7 \\3x + y - z &= 6\end{aligned}$$

Problem 4.5. Find the complex number z such that

$$(2 + 3i)z = \frac{39}{2}$$

Problem 4.6. Find the complex number z such that

$$(2 - 3i)z = 17 + 7i$$

Problem 4.7. Compute

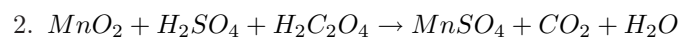
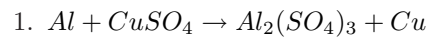
$$\frac{1}{2 + i}$$

Problem 4.8. Compute

$$\frac{1 + i}{-2 + 5i}$$

4.5 Problems II

Problem 4.9. Balance the following reactions.



Problem 4.10. The equations

$$2x + 7 = 3$$

$$bx - 10 = -2$$

have the same solution x . What is the value of b ?

Problem 4.11. The parabola with equation

$$y = x^2 + bx + c$$

passes through the points $\{(2, 3), (4, 3)\}$ What are b and c ?

Problem 4.12. The sum of three numbers is 20. The first is four times the sum of the other two. The second is seven times the third. What is the product of all three?

4.6 Problems III

Problem 4.13. Solve

$$\begin{aligned}x + y &= 1 \\x^2 + y^2 &= 2\end{aligned}$$

Problem 4.14. Solve

$$\begin{aligned}x + y &= -2 \\x^2 + y^2 &= 4\end{aligned}$$

Problem 4.15. Solve

$$\begin{aligned}2y &= x^2 \\4y &= x^3\end{aligned}$$

Problem 4.16. Solve

$$\begin{aligned}x &= y^2 \\x^2 - y^2 &= 2\end{aligned}$$

Problem 4.17. Solve

$$\begin{aligned}x^2 + y^2 &= 1 \\2x^2 - 3y^2 &= 3\end{aligned}$$

4.7 Problems IV

Problem 4.18. Find all 9 solutions to the 2 cubic equations

$$\begin{aligned}x^3 + 2y^3 - x - 2y &= 0 \\2x^3 - y^3 - 2x + y &= 0\end{aligned}$$

Problem 4.19. Consider the function $f(x) = 2^x$.

1. Use the two data points $f(0) = 1$ and $f(1) = 2$ to approximate the square root of 2 by finding a line that goes through those two points. How does this compare to the actual value? 1.414213562373.
2. Use the additional data point $f(2) = 4$ to find a quadratic approximation to estimate the square root of 2. How does this compare to the actual value?
3. Using more and more data points, will you ever get the exact value? Justify your answer.

Problem 4.20. Consider the function $f(x) = 2^x$.

1. Use the two data points $f(0) = 1$ and $f(1) = 2$ to approximate the cube root of 2 by finding a line that goes through those two points. How does this compare to the actual value? 1.25992104.
2. Use the additional data point $f(2) = 4$ to find a quadratic approximation to estimate the cube root of 2. How does this compare to the actual value?
3. Using more and more data points, will you ever get the exact value? Justify your answer.

Problem 4.21. Consider the function $f(x) = 2^x$.

1. Use the two data points $f(4) = 16$ and $f(5) = 32$ to approximate the square root of 2 by finding a line that goes through those two points. How does this compare to the actual value? 1.414213562373.
2. Use the additional data point $f(6) = 64$ to find a quadratic approximation to estimate the square root of 2. How does this compare to the actual value?
3. Using more and more data points, will you ever get the exact value? Justify your answer.

5 Friday

5.1 Project discussions and presentations.

6 Elliptic Curves

6.1 Project Description

Elliptic curves are curves represented by $y^2 = x^3 + ax^2 + bx + c$. They form the basis of what is known as elliptic curve cryptography. In this project you will explore some of the basic algebraic properties of these curves that allow them to be used for cryptography.

The operation you need is "addition" of points. Addition is in quotes, because it is a very special kind of addition which we will now define. Let P and Q be two points on your elliptic curve C . We know that two points define a line and any line intersects a cubic equation in three points. In other words, if we connect P, Q with a line, then it intersects the curve C in a third point. Call this third point $R = (x_3, y_3)$. Then we defined $P + Q = (x_3, -y_3)$. Note, that to compute $P + P$ we use the tangent line to C and P .

This "addition" on elliptic curves has some very nice properties, one of the most important is that there are points for which $P + P + \dots + P = P$. These points are called torsion points. In this project we will find formulas for this "addition" for specific and general curves and find some of these "torsion" points.

6.2 (Part 1):

Your first task in this project is to calculate some specific additions of points. Find $R = P + Q$ for the following curves. Recall the steps to do this:

1. Find the line that connects P, Q
2. Find the intersection points of this line with the elliptic curve (there will 3 of them)
3. Take this third point and define R to be the x value and negative the y value of this third point.

Example 6.1. Here is an example. Let C be defined by $y^2 = x^3 + 1$ and $P = (2, 3)$ and $Q = (0, 1)$. We compute as follows:

1. The line joining P and Q is given by $y = x + 1$.
2. The points of intersection of $y = x + 1$ and $y^2 = x^3 + 1$ are the solutions to

$$(x + 1)^2 = x^3 + 1$$

obtained by substituting $y = x + 1$ into $y^2 = x^3 + 1$. This equation simplifies to

$$x^3 - x^2 - 2x = 0.$$

This factors into

$$x(x - 2)(x + 1) = 0$$

since we know 0 and 2 are roots of the equation since they are points on the curve. So the third point of intersection is

$$(-1, 0).$$

3. So then $R = (-1, 0)$ since $-0 = 0$.

Calculate $P + Q$ for the following curves and points.

- (a) $C : y^2 = x^3 + 17, P = (2, 5), Q = (4, 9)$
- (b) $C : y^2 = x^3 - x + 4, P = (0, 2), Q = (-1, 2)$
- (c) $C : y^2 = x^3 - x + 1, P = (0, 1), Q = (-1, 1)$
- (d) $C : y^2 = x^3 - x + 9, P = (0, 3), Q = (-1, 3)$

6.3 (Part 2):

Now find a formula for $R = (x_3, y_3)$ in terms of $P = (x_1, y_1), Q = (x_2, y_2)$ for one of (a)-(d) from part 1. To do this, you will follow the same procedure as Part 1, but will have to solve for the variable x_3 where the three roots of the intersection equation are x_1, x_2 , and x_3 .

6.4 (Part 3):

Now find a formula for $P + P = R = (x_3, y_3)$ in terms of $P = (x_1, y_1)$ for the curve you used in Part 2. Note, that the slope of the tangent line to C at a point (x, y) is given by

$$m = \frac{3x^2 + 2ax + b}{2y}.$$

Caution: Do not just plug in $P + P$ into your formula from part 2. See if you can explain why that does not work.

6.5 (Optional Part 4):

Find some points of order 2, 3, or 4 for your curve.

- (a) Points of order 2 (2-torsion points) can be found by solving the equations $P + P = P$.
- (b) Points of order 3 can be found by solving the equation $P = -P$.
- (c) Points of order 4 (4-torsion points) can be found by solving the equation $P + P = -P$.

Justify why these equations are true.

7 Elliptic Curves Project 2

7.1 Project Description

Elliptic curves are curves represented by $y^2 = x^3 + ax^2 + bx + c$. They form the basis of what is known as elliptic curve cryptography. In this project you will explore some of the basic algebraic properties of these curves that allow them to be used for cryptography.

The operation you need is "addition" of points. Addition is in quotes, because it is a very special kind of addition which we will now define. Let P and Q be two points on your elliptic curve C . We know that two points define a line and any line intersects a cubic equation in three points. In other words, if we connect P, Q with a line, then it intersects the curve C in a third point. Call this third point $R = (x_3, y_3)$. Then we defined $P + Q = (x_3, -y_3)$. Note, that to compute $P + P$ we use the tangent line to C and P .

This "addition" on elliptic curves has some very nice properties, one of the most important is that there are points for which $P + P + \dots + P = P$. These points are called torsion points. In this project we will find formulas for this "addition" for specific and general curves.

7.2 (Part 1):

Your first task in this project is to calculate some specific additions of points. Find $R = P + Q$ for the following curves. Recall the steps to do this:

1. Find the line that connects P, Q
2. Find the intersection points of this line with the elliptic curve (there will 3 of them)
3. Take this third point and define R to be the x value and negative the y value of this third point.

Example 7.1. Here is an example. Let C be defined by $y^2 = x^3 + 1$ and $P = (2, 3)$ and $Q = (0, 1)$. We compute as follows:

1. The line joining P and Q is given by $y = x + 1$.
2. The points of intersection of $y = x + 1$ and $y^2 = x^3 + 1$ are the solutions to

$$(x + 1)^2 = x^3 + 1$$

obtained by substituting $y = x + 1$ into $y^2 = x^3 + 1$. This equation simplifies to

$$x^3 - x^2 - 2x = 0.$$

This factors into

$$x(x - 2)(x + 1) = 0$$

since we know 0 and 2 are roots of the equation since they are points on the curve. So the third point of intersection is

$$(-1, 0).$$

3. So then $R = (-1, 0)$ since $-0 = 0$.

Calculate $P + Q$ for the following curves and points.

(a) $C : y^2 = x^3 + 17, P = (2, 5), Q = (4, 9)$

(b) $C : y^2 = x^3 - x + 4, P = (0, 2), Q = (-1, 2)$

(c) $C : y^2 = x^3 - x + 1, P = (0, 1), Q = (-1, 1)$

(d) $C : y^2 = x^3 - x + 9, P = (0, 3), Q = (-1, 3)$

7.3 (Part 2):

Consider the family of elliptic curves of the form

$$y^2 = x^3 - x + m^2$$

where we are allowing m to vary.

Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two points on the curve $y^2 = x^3 - x + m^2$ for some constant m .

(a) Compute $R = P + Q$ in terms of x_1 and x_2 .

(b) Compute $R = P + P$ in terms of x_1 and x_2 .

Note, that for (b) you will need to know that the slope of the tangent line to the curve $y^2 = x^3 - x + m^2$ at a point (x, y) is given by

$$\frac{\Delta y}{\Delta x} = \frac{3x^2 - 1}{2y}.$$

7.4 (Part 3):

For $P = (0, m)$ and $Q = (-1, m)$ find formulas for

(a) $P + Q$

(b) $P + 2Q$

(c) $2P + Q$

(d) $P - Q$

(e) $3P$

What can you say about these points when m is an integer?

8 Solving Polynomial equations

8.1 Project Description

After arithmetic, solving equations is probably the piece of mathematics most used for applications. In other words, after you describe your situation through equations, you must then solve your equations to get your answer. In this project you will investigate solving polynomial equations. Specifically you will work with the quadratic equation and explicit formulas for solutions to cubic polynomials.

8.2 (Part 1):

Recall that the quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gives the two roots of $f(x) = ax^2 + bx + c$.

(a): Choose a quadratic polynomial at random. Find its two roots using the quadratic equation. Verify that $(x - x_0)$ divides your polynomial for x_0 one of the two roots from the quadratic equations. For example, choosing $x^2 - 1$ we have the two roots $x = -1, 1$. We can then check that $\frac{x^2-1}{x-1} = x + 1$.

(b): You will now verify the quadratic equation *always* gives you the roots. In other words, you will work symbolically with $f(x) = ax^2 + bx + c$ and show that

$$f\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) = 0$$

and

$$f\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = 0.$$

8.3 (Part 2):

We will start by solving cubic polynomials of the form

$$f(x) = x^3 + px - q$$

for some constants p and q .

(a): Choose some cubic polynomials of this form for which you know the roots.

(b): You will now derive a formula to determine the roots similar to the quadratic equation for quadratic polynomials. The steps below will guide you through algebra, but will leave the actual work to you.

We are working with the equation

$$x^3 + px - q.$$

We would like to factor this into a linear polynomial and a quadratic polynomial since the linear portion will give one of the roots and the other two can be found by using the quadratic equation on the quadratic portion.

We know that for any B we have

$$z^3 - B^3 = (z - B)(z^2 + Bz + B^2).$$

If we add a linear term

$$C(z - B)$$

to this we get

$$z^3 - B^3 + C(z - B) = (z - B)(z^2 + Bz + B^2 + C).$$

1. We need to find a B and C (in terms of p and q) that satisfy

$$x^3 - B^3 + C(x - B) = x^3 + px - q.$$

2. Now solve the two equations for B^3 in terms of p and q .

3. Trial and error leads to the solution

$$B = (q + \sqrt{p^3 + q^2})^{1/3} + (q - \sqrt{p^3 + q^2})^{1/3}$$

Compute B^3 to verify that this really does give you a solution.

4. Recall that we are working with

$$x^3 + px - q = (x - B)(x^2 + Bx + B^2 + C).$$

We have found one root of the cubic $x^3 + px - q$. (B). We need to find the other two.

Substitute the value of C back into the quadratic portion and use the quadratic equation find the other two solutions in terms of B .

(c): Check that this formula works for the cubics you created in (a).

8.4 (Part 3):

You will now find a formula to solve the general cubic. In other words, polynomials of the form

$$f(x) = x^3 + ax^2 + bx + c$$

for some constants a , b , and c .

(a): Choose some cubic polynomials of this form for which you know the roots.

(b): The steps below will guide you through algebra, but will leave the actual work to you.

We are starting with the equation

$$x^3 + ax^2 + bx + c.$$

1. Make the substitution

$$x = z - \frac{a}{3}.$$

You will end up with a cubic polynomial in the variable z which has no z^2 term.

2. Let p and $-q$ be the coefficients of the linear and constant terms respectively. In other words, you have

$$z^3 + pz - q = 0 \tag{1}$$

where p and q are in terms of a , b , and c . Notice that this is the form from Part 2.

3. Use Part 2 to find the three roots of equation (1).
4. Now you can substitute back to get x from z and the three roots of the original polynomial. Your end result should be equations for the 3 roots in terms of a , b , and c (the coefficients of the polynomial).

(c): Check that this formula works for the cubics you created in (a).