

# COMPLEX HYPERBOLIC IDEAL TETRAHEDRAL GROUPS.

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ABSTRACT. Tetrahedral groups are defined as groups of complex reflections on four planes, such that those planes form a tetrahedral configuration with vertices in the boundary of  $\mathbf{H}_{\mathbb{C}}^3$ . We prove that the complex tetrahedral groups that are complexifications of real groups (i.e. subgroups of  $PO(3,1)$ ) do not admit discrete deformations. To achieve this, the structure of the subgroups that stabilize the vertices is studied and it is proven that they correspond to certain discrete representations of abstract triangle groups  $\Delta$  in the Heisenberg group. These representations are studied and complete results are given for the case of  $\Delta$  been of Euclidean or spherical type. We also show an explicit representation of a hyperbolic triangle group in  $\mathcal{H}_5$ .

## 1. INTRODUCTION

In this work we try to understand the geometry of some of the most fundamental geometric objects in complex hyperbolic  $n$  dimensional space, namely, ideal tetrahedrons. (For  $n > 2$ .) By Mostow's rigidity theorem [13] any cocompact and discrete subgroup of isometries  $G$  that acts freely in  $\mathbf{H}_{\mathbb{C}}^n$  is rigid. In other words,  $G$  is the fundamental group of a manifold  $M \cong \mathbf{H}_{\mathbb{C}}^n/G$ , and  $G$  determines  $M$  uniquely up to an isometry. This theorem is not longer true in general for cusped manifolds. In [10] Goldman and Parker showed that discrete deformations of an ideal triangle group of reflections are possible in  $PU(n,1)$  (the isometry group of complex hyperbolic  $n$ -space,  $\mathbf{H}_{\mathbb{C}}^n$ ), even though that is not possible in  $PO(n,1)$  (the isometry group of real hyperbolic  $n$ -space). Motivated by that work we ask the question of whether deformations of other groups of reflections are possible. This paper partially answers this question in the negative for the simplest generalization of the class of groups studied by Goldman and Parker, namely *complex ideal tetrahedral groups*.

There are other questions that remain to be addressed. For example to obtain the complete classification of all discrete tetrahedral groups. Some work has already been done on that direction. In [6] the possible cusps for tetrahedral groups are classified.

Consider a set of four points  $\mathcal{C} = \{Q_1, Q_2, Q_3, Q_4\} \subset \partial\mathbf{H}_{\mathbb{C}}^n$  such that no three of them are contained in the boundary of a one dimensional complex linear subspace of  $\mathbf{H}_{\mathbb{C}}^3$ . For any three points in  $\mathcal{C}$ , construct the unique complex two linear subspace in  $\mathbf{H}_{\mathbb{C}}^n$  that contains those points in its boundary. Associated to each of these complex planes there is a *complex reflection*, that is to say, an elliptic isometry of order two that fixes the plane. The group generated by the four possible reflections will be called the *ideal tetrahedral group associated to the configuration  $\mathcal{C}$* .

Using the diagram shown in figure 1 as a reference we fix some notation. Name  $\rho_i$  the reflection on the complex two plane that contains  $\mathcal{C} - \{Q_i\}$ . The map  $\rho_i\rho_j$

will be denoted by  $\nu_{ij}$ . This map fixes the complex line through  $Q_k$  and  $Q_l$ , where  $\{i, j, k, l\} = \{1, 2, 3, 4\}$ . Using this notation we write from now on:

$$K = \langle \rho_1, \rho_2, \rho_3, \rho_4 \rangle \subset PU(n, 1)$$

for the ideal tetrahedral group associated to the configuration  $\mathcal{C}$ .

The first observation one can do is that any four points in the boundary of  $\mathbf{H}_{\mathbf{C}}^n$  are always contained in a three dimensional linear subspace. This linear subspace can be considered as an immersed copy of  $\mathbf{H}_{\mathbf{C}}^3$  in  $\mathbf{H}_{\mathbf{C}}^n$ . Therefore it is enough to restrict the discussion to  $\mathbf{H}_{\mathbf{C}}^3$  as we will do in all this paper.

The union of the four complex planes determined by the configuration of points  $\mathcal{C}$  will be called the *complex tetrahedron* or *complex tetrahedron associated to  $\mathcal{C}$* .

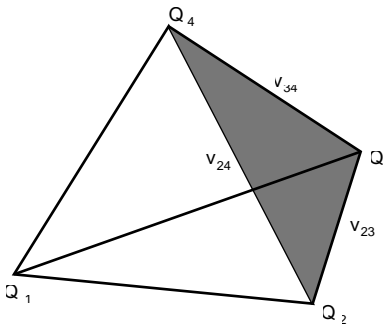


FIGURE 1. Representation of a complex ideal tetrahedron with vertices in  $\mathcal{C}$

An analogous construction can be done in the case of real hyperbolic three space (see for example [16] and [3]), the resulting group, which we call  $K_{\mathbf{R}}$  is a subgroup of  $PO(3, 1)$  that depends on one complex parameter  $z$ . As it is well known,  $\partial\mathbf{H}_{\mathbf{R}}^3 \cong \hat{\mathbf{C}}$ —the Riemann sphere—and any four distinct points in  $\partial\mathbf{H}_{\mathbf{R}}^3$  can always be mapped to  $0, 1, \infty$  and  $z \in \mathbf{C}$  by an element of  $PO(3, 1)$ .

**Theorem 1.1.** *The group  $K_{\mathbf{R}}$  is discrete if and only if  $0, 1$ , and  $z$  are the vertices of a Euclidean triangle in  $\mathbf{C}$  with interior angles  $(\pi/2, \pi/4, \pi/4)$ ,  $(\pi/3, \pi/3, \pi/3)$  or  $(\pi/2, \pi/4, \pi/4)$ .*

For a proof of the previous theorem see, for example, [15, Theorem 7.1.3]. In other words, up to isometries of  $\mathbf{H}_{\mathbf{R}}^3$ , there are only three discrete real hyperbolic triangle groups. By the natural inclusion  $PO(3, 1) \subset PU(3, 1)$ , the group  $K_{\mathbf{R}}$  also acts in  $\mathbf{H}_{\mathbf{C}}^3$ , we call this class of groups *real tetrahedral*. Geometrically, the inclusion  $K_{\mathbf{R}} \subset PU(3, 1)$  corresponds to assigning every generating reflection  $\xi$  of  $K_{\mathbf{R}}$ , the complex reflection  $\rho_{\xi}$  that fixes the unique complex plane that contains the real plane fixed by  $\xi$ . The *complexified group*  $K$  will be then the group generated by the reflections  $\rho_{\xi}$ . We prove that these groups do not admit deformations in  $PU(3, 1)$ :

**Theorem 1.2.** *Let  $K \subset PO(3, 1) \subset PU(3, 1)$  be a real tetrahedral group, then it does not admit any deformations in  $PU(3, 1)$  that are not induced by isometries.*

To study the group  $K$  it is necessary first to concentrate on the subgroups generated by three of the reflections that generate  $K$ . Define the group  $G$  as the group in  $PU(3, 1)$  generated by  $\rho_1, \rho_2$  and  $\rho_3$ . Since  $PU(3, 1)$  is transitive in  $\mathbf{H}_{\mathbf{C}}^3$ ,

it can be assumed without losing generality that  $Q_4 = p_\infty$ . This is the point of intersection in  $\partial\mathbf{H}_\mathbb{C}^3$  of the three complex two-planes fixed by the reflections  $\rho_i$ . Call the corresponding two-chains  $P_1, P_2$  and  $P_3$ , respectively.

The group  $G$  stabilizes  $p_\infty$ , hence it can be considered as a subgroup of the symmetry group of  $\mathcal{H}_5$  (that is identified with  $\partial\mathbf{H}_\mathbb{C}^3 - \{p_\infty\}$ ). The same notation will be used for an element of  $G$  acting in  $\mathbf{H}_\mathbb{C}^3$  and for the induced map on Heisenberg space. The chains  $P_i$  are vertical in  $\mathcal{H}_5$ , therefore the reflections  $\rho_i$  all have the form  $\rho = T \circ S$ , where  $T$  is a Heisenberg translation, and  $S$  is a complex reflection about a vertical two-chain through the origin of  $\mathcal{H}_5$ . This characterization of the generators of  $G$  shows that  $G$  is a subgroup of the Heisenberg isometries ( $\text{Isom}(\mathcal{H}_5)$ ). If all the two-chains coincide then  $K = G \cong \mathbf{Z}/2\mathbf{Z}$ , in that case we will say that the group  $K$  (or  $G$ ) is almost trivial.

A necessary condition for  $K$  to be discrete is that  $G$  is also discrete, and hence all the subgroups of  $G$  should be discrete. We will say that  $G$  is *discrete on dihedral subgroups* if the subgroups  $\langle \rho_i, \rho_j \rangle_{i \neq j}$  are discrete in  $\text{Isom}(\mathcal{H}_5)$ . The following theorem (that will be proved on section 3) relates all these facts.

**Theorem 1.3.** *If the group  $G$  is discrete on dihedral groups then  $G$  is the representation of a triangle group  $\Delta$ . If the group  $G$  is finite then it is isomorphic to  $\mathbf{Z}/2\mathbf{Z}$ , in that case we will say that the representation is almost trivial. In particular, representations of spherical triangle groups are almost trivial.*

Let  $H$  be an abstract group and  $W$  a subspace of  $\mathbf{C}^{n-1}$ , a representation  $\bar{\phi} : H \rightarrow \text{Isom}(\mathbf{C}^{n-1})$  is said to be *induced* by a representation  $\tilde{\phi} : H \rightarrow \text{Isom}(W)$  if  $\tilde{\phi} \circ p = p \circ \bar{\phi}$ , where  $p : \mathbf{C}^n \rightarrow W$  is the orthogonal projection. A representation  $\phi : H \rightarrow \text{Isom}(\mathcal{H}_{2n-1})$  is said to be *vertical* if there exists a representation  $\bar{\phi} : H \rightarrow \text{Isom}(\mathbf{C}^{n-1})$  so that  $\bar{\phi} = \Pi_V \circ \phi$  and  $\ker(\phi) = \ker(\bar{\phi})$ . A vertical representation  $\phi$  will be called *real* if there exist a real  $(n-1)$  dimensional subspace  $W \subset \mathbf{C}^{n-1}$  such that  $\Pi_V \circ \phi$  is induced by its restriction to  $W$ .

**Theorem 1.4.** *If  $\Delta$  is an abstract triangle group of Euclidean type, the representation  $\phi : \Delta \rightarrow G \subset \text{Isom}(\mathcal{H}_5)$  is discrete if and only if it is real.*

It is important to observe that although the group  $K$  is determined by the position of four points in  $\partial\mathbf{H}_\mathbb{C}^3$ , the group  $G$  is only determined by the three vertical 1-chains defined by:  $c_{ij} = P_i \cap P_j$ . These three 1-chains are the boundaries of the complex geodesics that join the point at infinity ( $p_\infty$ ) with the other three points. We will call the union of these chains  $\mathcal{L}$ . Moreover, since each of the 2-chains  $P_i$  is vertical, its vertical projection onto  $\mathbf{C}^2$  will be a complex line  $l_i$ , and the vertical projections of the 1-chains  $c_{ij}$  is the point  $p_{ij}$  of  $\mathbf{C}^2$  defined by the intersection of the lines  $l_i$  and  $l_j$ , respectively. Therefore  $\Pi_V(\mathcal{L}) = \{p_{12}, p_{13}, p_{23}\}$ , and theorem 1.4 can be rephrased as follows:

**Theorem 1.5.** *If  $\Delta$  is an abstract triangle group of Euclidean type, the representation  $\phi : \Delta \rightarrow G \subset \text{Isom}(\mathcal{H}_5)$  is discrete if and only if  $\Pi_V(\mathcal{L})$  is contained in a totally real subspace of  $\mathbf{C}^2$ .*

Theorem 1.2 can now be proven by applying this theorem on each of the vertices of the complex tetrahedron associated to  $\mathcal{C}$ . As a result of the study of representations of triangle groups we will also prove:

**Theorem 1.6.** *There exists a (non real and non faithful) representation  $\phi : \Delta \rightarrow \text{Isom}(\mathcal{H}_5)$  of a hyperbolic triangle group  $\Delta = \Delta_{(4,4,4)}$ .*

John Parker has noticed that this is a subgroup of the matrix group  $GL(4, \mathbf{Z}[i])$ , and therefore this representation is discrete. The associated tetrahedral groups form a 2 dimensional family of groups, there is evidence that the representation of the associated tetrahedral group is rigid (that is all representations are conjugate by an element of  $PU(n, 1)$ ). This result together with the complete classification of all possible representations of hyperbolic triangle groups will be treated separately in a future paper.

This paper is organized as follows. In section 2 we introduce the basic definitions and notations of complex hyperbolic space and its boundary, the Heisenberg group, and Cartan invariant. We also review some basic facts on triangle groups, the geometry of  $SU(2)$  and of  $\mathbf{C}^2$ . Section 3 gives an explicit form for the group  $G$ , and proves that  $G$  is a representation of a triangle group. In section 4 we prove that  $G$  can not be finite unless the group  $K$  is isomorphic to  $\mathbf{Z}/2\mathbf{Z}$ . Section 5 is devoted to the study of discrete representations of a (hyperbolic or Euclidean) triangle group in  $\mathcal{H}_5$ , we specialize to the case of Euclidean triangle groups and prove that in that case the representation must be real. Finally in section 6 we return to tetrahedral groups and, using the geometry of  $\mathbf{H}_{\mathbf{C}}^3$ , prove theorem 1.2.

The main results of this paper are part of my thesis, I would like to thank my advisor, Domingo Toledo, for all his help and support. I would also like to thank, A. Bernasconi, N. Brady, and M. Kapovich for helpful discussions while this paper was been developed, and J. Parker who read earlier versions of this manuscript and gave me very useful suggestions to improve the presentation of the paper.

## 2. PRELIMINARIES

The material of 2.1, 2.2, 2.3 and 2.4 is a review of the geometry of complex hyperbolic space and its boundary, the Heisenberg space. The main reference for these topics is [9], see also [5], [11], [14] and [17]. In 2.5 we write explicit formulas for complex reflections and the angle of holomorphy of a 2-plane in  $\mathbf{C}^2$ . Triangle groups and quaternions are reviewed in 2.6 and 2.7, respectively.

**2.1. Complex Hyperbolic Space.** Let us define the space  $\mathbf{C}^{n,1}$  as the vector space  $\mathbf{C}^{n+1}$  with the Hermitian bilinear form  $\langle Z, W \rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n - z_{n+1} \bar{w}_{n+1}$  for vectors  $Z = (z_1, \dots, z_{n+1})$  and  $W = (w_1, \dots, w_{n+1})$ . A vector  $v \in \mathbf{C}^{n,1}$  is said to be *negative* (respectively *positive* or *null*) if  $\langle v, v \rangle < 0$  (respectively  $\langle v, v \rangle > 0$  or  $\langle v, v \rangle = 0$ ). Similarly, since the bilinear form is homogeneous, equivalence classes in  $\mathbf{P}(\mathbf{C}^{n,1})$  are divided in negative, positive or null. *Complex Hyperbolic  $n$  space*, denoted  $\mathbf{H}_{\mathbf{C}}^n$ , is defined to be the set of negative equivalence classes in  $\mathbf{P}(\mathbf{C}^{n,1})$ .

We fix the notation  $\langle\langle z, w \rangle\rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n$  for the usual Hermitian pairing of vectors  $z$  and  $w$  in  $\mathbf{C}^n$ .

Complex hyperbolic  $n$  space admits a *Bergmann* metric that is  $PU(n, 1)$  invariant and of constant holomorphic sectional curvature  $-1$ . The group of holomorphic isometries of  $\mathbf{H}_{\mathbf{C}}^n$  is  $PU(n, 1)$ . The isometries with a fixed point inside  $\mathbf{H}_{\mathbf{C}}^n$  are called *elliptic*. Those that fix a unique pair of points at the boundary are called *loxodromic*. And the isometries that fix a unique point at the boundary are called *parabolic*.

Let  $V$  be a  $k+1$  complex dimensional subspace with nontrivial intersection with the cone  $C$  of negative vectors in  $\mathbf{C}^{n,1}$ . The projection of  $V \cap C$  to  $\mathbf{H}_{\mathbf{C}}^n$  defines a complex linear  $k$ -dimensional subspace of  $\mathbf{H}_{\mathbf{C}}^n$ . Similarly, given a  $k+1$  real subspace

$W$  of  $\mathbf{C}^{n,1}$  with nontrivial intersection with the cone  $C$ , the projection of  $W \cap C$  to  $\mathbf{H}_{\mathbf{C}}^n$  defines a real  $k$ -subspace.

Real subspaces of dimension one are geodesics. Any real  $k$ -subspace ( $k > 1$ ) is a totally geodesic real  $k$  dimensional smooth manifold of  $\mathbf{H}_{\mathbf{C}}^n$  whose sectional curvature is constant equal to  $-1/4$  (this of course depends on the normalization  $K_H = -1$  for the holomorphic sectional curvature). Note also that *complex geodesics* (complex linear one-dimensional subspaces) are subspaces of constant curvature  $-1$ .

2.1.1. *Reflections in  $\mathbf{H}_{\mathbf{C}}^n$ .* Let  $F$  be a complex codimension one subspace of  $\mathbf{C}^{n,1}$ , such that the Hermitian form restricted to  $F$  is non degenerate. Then there is a positive vector  $f \in \mathbf{C}^{n,1}$  such that  $F = f^\perp$ . A reflection  $\varrho$  of order two around the complex  $n - 1$ -space defined by  $f$  is defined by the formula:

$$\varrho_F(Z) = Z - 2 \frac{\langle Z, f \rangle}{\langle f, f \rangle} f.$$

In this paper we will call the reflections of order two, for short, *reflections*. Any two reflections that fix a point  $p_\infty \in \partial \mathbf{H}_{\mathbf{C}}^n$  are conjugate to each other by a reflection fixing  $p_\infty$ . In particular all reflections fixing  $p_\infty = (0, \dots, 0, -1, 1)^t$ , are conjugate to the reflection defined by the matrix

$$A = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} \in U(n, 1).$$

2.1.2. *Horospherical Coordinates.* Goldman and Parker ([11], see also [9]) define *horospherical coordinates* for points  $Z$  in  $\overline{\mathbf{H}}_{\mathbf{C}}^n$  via the correspondence

$$\begin{aligned} \mathbf{C}^{n-1} \rtimes \mathbf{R} \times \mathbf{R}_{\geq 0} &\rightarrow \overline{\mathbf{H}}_{\mathbf{C}}^n - \{p_\infty\} \\ (\zeta, v, u) &\mapsto Z \end{aligned}$$

here  $(\zeta, v, u)$  are called the *horospherical coordinates* of the corresponding point  $Z$  in  $\overline{\mathbf{H}}_{\mathbf{C}}^n$ . The inverse of this correspondence is given by

$$\begin{aligned} \zeta &= \frac{Z'}{Z_n + Z_{n+1}}, \quad v = \Im \left( \frac{Z_n - Z_{n+1}}{Z_n + Z_{n+1}} \right) \\ u &= -\frac{\langle Z, Z \rangle}{|Z_n + Z_{n+1}|^2}. \end{aligned}$$

Level sets of the function  $u$  define horospheres (in the sense of [2]) centered at the point  $p_\infty$ . The coordinates  $(\zeta, v)$  represent the parabolic transformation defined by the matrix:

$$(1) \quad H(\zeta, v) = \begin{bmatrix} I_{n-1} & \zeta & \zeta \\ -\zeta^* & 1 - \frac{1}{2}(\|\zeta\|^2 - iv) & -\frac{1}{2}(\|\zeta\|^2 - iv) \\ \zeta^* & \frac{1}{2}(\|\zeta\|^2 - iv) & 1 + \frac{1}{2}(\|\zeta\|^2 - iv) \end{bmatrix}$$

that takes the point  $p_o = (0', 1, 1)^t$  with horospherical coordinates  $(0, 0, 0)$  to the point with horospherical coordinates  $(\zeta, v, 0)$ .

The stabilizer of  $p_\infty$  contains also the group of elliptic transformations given by

$$(2) \quad R_U = \begin{bmatrix} U & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where  $U \in U(n-1)$ . The transformations that are compositions of elliptic (2) and parabolic (1) transformations are called *ellipto-parabolic*.

**2.2. The Boundary of  $\mathbf{H}_{\mathbf{C}}^n$  and the Heisenberg Group.** The set of automorphisms of the form (1) form a nilpotent subgroup of  $PU(n,1)$  that acts simply transitively on the boundary minus  $p_{\infty}$ , we will call this group the *Heisenberg group* (denoted by  $\mathcal{H}_{2n-1}$ ) and we will identify it with the points in  $\partial\mathbf{H}_{\mathbf{C}}^n - \{p_{\infty}\}$ . The group multiplication is given by the formula:

$$(\zeta, v) \cdot (z, u) = (\zeta + z, v + u + 2 \operatorname{Im}\langle\zeta, z\rangle).$$

**2.2.1. Automorphisms and Isometries.** We distinguish three subgroups of automorphisms in  $\mathcal{H}_{2n-1}$ :

- Heisenberg dilations, defined for every number  $s > 0$  by  $D_s(\zeta, v) = (s\zeta, s^2v)$ .
- Heisenberg translations, defined for every  $(z, u) \in \mathcal{H}_{2n-1}$  via the group multiplication by  $T_{(z,u)}(\zeta, v) = (z, u) \cdot (\zeta, v)$ .
- Heisenberg rotations,  $A \cdot (\zeta, v) = (A\zeta, v)$ , for  $A \in U(n-1)$ .

The group generated by these transformations is the group of automorphisms of  $\mathcal{H}_{2n-1}$  (denoted by  $\operatorname{Aut}(\mathcal{H}_{2n-1})$ ) and corresponds to the stabilizer of  $p_{\infty}$  in  $PU(n,1)$ . Heisenberg translations of the form  $T_{(0,v)}$  are called *vertical* and they compose the center of  $\mathcal{H}_{2n-1}$ . The Heisenberg group is in fact a central extension of  $\mathbf{C}^{n-1}$  by  $\mathbf{R}$ , and hence also  $\operatorname{Aut}(\mathcal{H}_{2n-1})$ . We will write  $\mathcal{H}_{2n-1} \cong \mathbf{C}^n \rtimes \mathbf{R}$  and  $\operatorname{Aut}(\mathcal{H}_{2n-1}) \cong \operatorname{Aut}(\mathbf{C}^n) \rtimes \mathbf{R}$ .

The *Cygan metric* is defined in  $\mathcal{H}_{2n-1}$  by

$$d_C((\zeta, v), (\xi, u)) = |(\zeta, v)^{-1} \cdot (\xi, u)|,$$

where

$$|(\zeta, v)| = (|\zeta|^4 + v^2)^{1/4}.$$

The subgroup of  $\operatorname{Aut}(\mathcal{H}_{2n-1})$  that leaves this metric invariant is called the set of isometries (denoted  $\operatorname{Isom}(\mathcal{H}_{2n-1})$ ) of  $\mathcal{H}_{2n-1}$ . This group is isomorphic to the semidirect product of  $\mathcal{H}_{2n-1}$  with  $U(2)$ :  $\operatorname{Isom}(\mathcal{H}_{2n-1}) \cong \mathcal{H}_{2n-1} \rtimes U(2)$ .

**2.2.2. Vertical and Geodesic Projections.** The real geodesic that joins the point in  $\mathbf{H}_{\mathbf{C}}^n$  with horospherical coordinates  $(\zeta, v, u)$  and the point at infinity,  $p_{\infty}$  is  $t \mapsto (\zeta, v, t)$ ,  $t > 0$ . Allowing  $t$  to attain the value 0, one gets to the boundary of  $\mathbf{H}_{\mathbf{C}}^n$ . Then the point with horospherical coordinates  $(\zeta, v, 0) \in \partial\mathbf{H}_{\mathbf{C}}^n$  will be written simply as  $(\zeta, v)$ . The projection of  $\mathbf{H}_{\mathbf{C}}^n$  to  $\partial\mathbf{H}_{\mathbf{C}}^n$  along these geodesics, called the *geodesic projection*, which in horospherical coordinates is given by:

$$(3) \quad \begin{aligned} \Pi_g : \mathbf{H}_{\mathbf{C}}^n &\rightarrow \mathcal{H}_{2n-1} \\ (\zeta, v, u) &\mapsto (\zeta, v). \end{aligned}$$

For more details see [9] and [11].

**2.2.3. Contact Structure on  $\partial\mathbf{H}_{\mathbf{C}}^n$ .** The boundary of complex hyperbolic space  $\partial\mathbf{H}_{\mathbf{C}}^n \subset \mathbf{P}(\mathbf{C}^{n,1})$  admits a canonical CR structure. A CR structure is a codimension one plane field  $\mathcal{E}$  of the tangent bundle  $T(\partial\mathbf{H}_{\mathbf{C}}^n)$ , carrying a complex structure satisfying certain integrability conditions (for details see [9] and [12]). In  $M = \partial\mathbf{H}_{\mathbf{C}}^n$  the CR structure  $\mathcal{E} = \{E_x\}_{x \in M}$ , is defined by:

$$E_x = T_x M \cap \mathbf{J}T_x M$$

This CR structure is also a *contact structure*. A contact structure of  $M^{2n-1}$  is a nondegenerate hyperplane field  $\mathcal{E}$  (see for example [1]) in the sense that  $\omega \wedge d\omega^{n-1}$  is a volume form for  $M$ , where  $\omega$  is a *calibration* for  $\mathcal{E}$ . A calibration for  $\mathcal{E}$  is a one form  $\omega : TM \rightarrow \mathbf{R}$  such that  $\ker \omega = \mathcal{E}$ . Calibrations are not unique but given two different calibrations  $\omega_1$  and  $\omega_2$  of  $\mathcal{E}$ , there is a nonzero function  $f : M \rightarrow \mathbf{R}$  such that  $\omega_1 = f\omega_2$ . A calibration for  $\partial\mathbf{H}_{\mathbf{C}}^n - \{p_\infty\} \cong \mathcal{H}_{2n-1}$  in terms of horospherical coordinates of the boundary  $(\zeta, v)$  is given by

$$(4) \quad \omega(\zeta, v) = dv + 2 \operatorname{Im}\langle d\zeta, \zeta \rangle.$$

**2.3. Chains and  $\mathbf{R}$ -spheres.** The boundary of a linear complex subspace of dimension  $k$  in  $\mathbf{H}_{\mathbf{C}}^n$  is called a *k-chain* (or simply a *chain* if  $k = 1$ ). The  $k$ -chains containing  $p_\infty$  are called *vertical*. In Heisenberg coordinates they are preimages under the vertical projection  $(\Pi_V)$  of complex affine  $k - 1$ -subspaces of  $\mathbf{C}^{n-1}$ . The vertical projection to  $\mathbf{C}^{n-1}$  of a non infinite  $k$ -chain is a  $(2k - 1)$ -dimensional sphere.

The boundary of a totally real  $k$ -subspace ( $k > 1$ ) in  $\mathbf{H}_{\mathbf{C}}^n$  is called an  $\mathbf{R}^k$ -sphere (or  $\mathbf{R}$ -circle if  $k = 2$ ). Note that an  $\mathbf{R}^k$ -sphere is a smooth manifold of real dimension  $k - 1$ . A  $\mathbf{R}^k$ -sphere that contains  $p_\infty$  is called infinite. In particular, in  $\mathbf{H}_{\mathbf{C}}^3$  the  $\mathbf{R}^3$ -sphere corresponding to  $\mathbf{H}_{\mathbf{R}}^3$  (the real points of  $\mathbf{H}_{\mathbf{C}}^3$ ) is an infinite  $\mathbf{R}^3$ -sphere  $\mathcal{S}$ . In horospherical coordinates

$$\mathcal{S} = \{(\zeta, v) \in \mathcal{H}_5 \mid \zeta \in \mathbf{R}^2 \text{ and } v = 0\}.$$

If  $P \subset \mathbf{H}_{\mathbf{C}}^n$  is an  $\mathbf{R}^k$ -plane, then the  $\mathbf{R}$ -sphere  $\partial P \subset \partial\mathbf{H}_{\mathbf{C}}^n$  is CR-horizontal with respect to the canonical CR-structure on  $\partial\mathbf{H}_{\mathbf{C}}^n$ .

**Theorem 2.1.** *Let  $L^k \subset \mathbf{H}_{\mathbf{C}}^n$  be a  $\mathbf{C}^k$ -plane and  $\partial L \subset \partial\mathbf{H}_{\mathbf{C}}^n = \mathcal{H}_{2n-1} \cup \{p_\infty\}$  its bounding  $k$ -chain. Let  $\Pi_V : \mathcal{H}_{2n-1} \rightarrow \mathbf{C}^{n-1}$  be the vertical projection.*

- i) *If  $\partial L$  is vertical then  $\Pi_V(\partial L - \{p_\infty\})$  is a  $\mathbf{C}$ -affine subspace of  $\mathbf{C}^{n-1}$  of dimension  $k - 1$  and  $\partial L - \{p_\infty\} = \Pi_V^{-1}(\Pi_V(\partial L - \{p_\infty\}))$ .*
- ii) *If  $\partial L$  is finite, then  $\Pi_V$  maps  $\partial L$  bijectively onto a Euclidean sphere in  $\mathbf{C}^{n-1}$  of dimension  $2k - 1$ .*
- iii) *Two  $\mathbf{C}^k$ -chains having the same vertical projection differ by a vertical translation.*

**Theorem 2.2.** *Let  $S \subset \partial\mathbf{H}_{\mathbf{C}}^n = \mathcal{H}_{2n-1} - \{p_\infty\}$  be an  $\mathbf{R}^k$ -sphere containing  $p_\infty$ . Then  $S - \{p_\infty\}$  is an affine subspace in  $\mathcal{H}_{2n-1}$  such that the vertical projection  $\Pi_V : \mathcal{H}_{2n-1} \rightarrow \mathbf{C}^{n-1}$  maps  $S - \{p_\infty\}$  injectively onto an affine totally real subspace of  $\mathbf{C}^{n-1}$ .*

For proofs of these theorems see [9].

Let  $L$  be a complex geodesic in  $\mathbf{H}_{\mathbf{C}}^2$ , and let  $\iota_L$  be the inversion (i.e., complex reflection of order two) around  $L$ . Let  $c$  be the corresponding chain:  $c = \partial L$ . Following Goldman ([9]) define the center of a finite chain  $c \subset \mathcal{H}_2$  to be the point  $O_c = \iota_L(p_\infty)$ . The radius of the chain is defined to be the radius of the projected circle  $\Pi_V(c) \subset \mathbf{C}$ . The radius and center define a chain  $c$  in a unique way. A chain with radius  $r_0$  and center  $O_c = (\zeta_0, v_0)$  is defined by the equations:

$$(5) \quad \|\zeta - \zeta_0\| = r_0, \quad v = v_0 - 2 \operatorname{Im}\langle \zeta, \zeta_0 \rangle.$$

**2.4. Cartan Invariant.** Following Élie Cartan [4] given a set of three points  $x = (x_1, x_2, x_3)$  in  $\partial\mathbf{H}_{\mathbf{C}}^n$  define the invariant  $\mathbf{A}(x)$  by:

$$\mathbf{A}(x_1, x_2, x_3) = \arg(-\langle X_1, X_2, X_3 \rangle),$$

where  $X_i$  are null vectors representing the points  $x_i$  and

$$\langle X_1, X_2, X_3 \rangle = \langle X_1, X_2 \rangle \langle X_2, X_3 \rangle \langle X_3, X_1 \rangle.$$

This invariant satisfies the following properties:

- i.  $-\pi/2 \leq \mathbf{A}(x) \leq \pi/2$ ;
- ii.  $\mathbf{A}(x) = 0 \Leftrightarrow x_1, x_2, x_3$  lie on an  $\mathbf{R}$ -circle;
- iii.  $\mathbf{A}(x) = \pm\pi/2 \Leftrightarrow x_1, x_2, x_3$  lie on a chain;
- iv. If  $g \in PU(n, 1)$ , then

$$\mathbf{A}(g(x_1), g(x_2), g(x_3)) = \mathbf{A}(x_1, x_2, x_3);$$

- v. if  $\Sigma_{12}$  is the unique complex geodesic joining  $x_1$  and  $x_2$  and  $\Pi_{12} : \mathbf{H}_{\mathbf{C}}^n \rightarrow \Sigma_{12}$  is the orthogonal projection, then  $\mathbf{A}(x_1, x_2, x_3) = \text{Area}(\Delta)/2$ , where  $\Delta$  is the hyperbolic triangle in  $\Sigma_{12}$  with vertices  $x_1, x_2$  and  $\Pi_{12}(x_3)$ .

By the following theorem of D. Toledo (see [9] and [17])  $\mathbf{A}$  is a cocycle.

**Theorem 2.3.** *Let  $x_1, x_2, x_3, x_4 \in \partial\mathbf{H}_{\mathbf{C}}^n$ , then*

$$\mathbf{A}(x_1, x_2, x_3) - \mathbf{A}(x_2, x_3, x_4) + \mathbf{A}(x_3, x_4, x_1) - \mathbf{A}(x_4, x_1, x_2) = 0.$$

**Lemma 2.4.** *Identify  $\partial\mathbf{H}_{\mathbf{C}}^3 - p_{\infty}$  with  $\mathcal{H}_5$ . Let  $p_0$  and  $p$  be the points in  $\mathcal{H}_5$  with horospherical coordinates  $((0, 0), 0)$  and  $(\zeta, 0)$ , respectively ( $\zeta \in \mathbf{C}^2$ ). Then  $\mathbf{A}(p_{\infty}, p_0, p) = \arg(\|\zeta\|^2 + iv)$ .*

*Proof.* The points  $p_{\infty}, p_0$  and  $p$  correspond to the null vectors

$$P_{\infty} = \begin{bmatrix} 0' \\ -1 \\ 1 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 0' \\ 1 \\ 1 \end{bmatrix}, \quad P = \begin{bmatrix} 2\zeta \\ 1 - \|\zeta\|^2 + iv \\ 1 + \|\zeta\|^2 - iv \end{bmatrix}$$

respectively, with  $\zeta, 0' \in \mathbf{C}^2$ . The Cartan invariant is given by:

$$\begin{aligned} \mathbf{A} &= \mathbf{A}(p_{\infty}, p_0, p) = \arg(-\langle P_{\infty}, P_0, P \rangle) \\ &= \arg(-\langle P_{\infty}, P_0 \rangle \langle P_0, P \rangle \langle P, P_{\infty} \rangle) = \arg(\|\zeta\|^2 + iv). \end{aligned}$$

Another, more geometric, proof can be deduced from the following fact (see [9, pp ??]):

Let  $\Sigma \subset \mathbf{H}_{\mathbf{C}}^3$  be the complex geodesic  $0 \times \mathbf{H}_{\mathbf{C}}^1 \subset \mathbf{H}_{\mathbf{C}}^3$  bounded by the chain that corresponds to the vertical axis  $\mathbf{V} = \{(0, 0)\} \times \mathbf{R}$  in  $\mathcal{H}_5$ . Then the expression  $\Upsilon(\zeta, v) = \|\zeta\|^2 - iv$  represents the composition of the orthogonal projection  $\Pi_{\Sigma} : \mathbf{H}_{\mathbf{C}}^3 \rightarrow \Sigma$  with the embedding  $\Sigma \hookrightarrow \mathbf{C}$  as the right half plane  $\mathfrak{h}$ .

The half plane  $\mathfrak{h}$  has a Poincaré metric given by  $ds = \frac{|dz|}{x}$ . By definition (see 2.4) the Cartan invariant  $\mathbf{A} = \mathbf{A}(p_{\infty}, p_0, p)$  is equal to half the signed area of the hyperbolic triangle in  $\mathfrak{h}$  with vertices  $\infty, 0$  and  $w = \Upsilon(p)$ . Denote by  $\Delta_w$  the ideal triangle with vertices  $0, \infty$  and  $w$ . And denote by  $[x, y]$  the hyperbolic geodesic segment with end points  $x$  and  $y$ .

By definition, we have:

$$(6) \quad \mathbf{A} = \frac{1}{2} \int_{\Delta_w} \frac{dz}{x} = \pm \frac{1}{2} (\pi - \alpha - \beta - \gamma).$$

The angles  $\alpha$ ,  $\beta$  and  $\gamma$  are the interior angles of  $\Delta_w$  at  $\infty$ ,  $0$  and  $w$ , respectively. (Compare Figure 2.) The angles  $\alpha$  and  $\beta$  are zero, because  $0$  and  $\infty$  are ideal points.

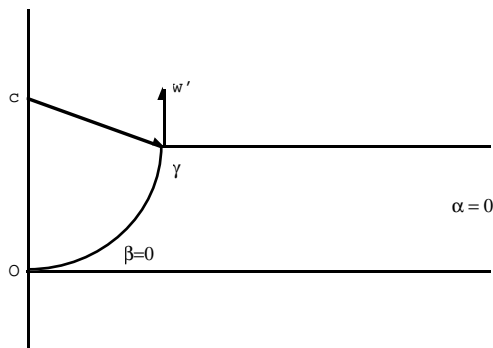


FIGURE 2. Hyperbolic triangle on  $\Sigma$  with vertices  $\infty$ ,  $0$  and  $w = \Upsilon(p)$ .

To compute  $\gamma$  observe first that if  $\arg w = 0$  then  $w$  would lie on the geodesic segment  $[0, \infty]$ . In that case  $d = \pi$  and  $\mathbf{A} = 0$ . Suppose that  $\arg w \neq 0$ , and from now on refer to Figure 2. The hyperbolic geodesic segment  $[0, w]$  is an arc of the circle  $|z - c| = |c|$ , with center  $c = i|w|^2/(2\Im w)$ .

The tangent vector to this circle at  $w$  is the vector  $w'$ , which is orthogonal to  $v = w - c$ . More precisely  $w' = iv = iw - ic$ . The angle  $d$  between the segments  $[0, w]$  and  $[w, \infty]$  is equal to  $d = -\arg(-w')$ . Assume first that  $\arg w > 0$ . Replacing  $w'$  in the previous formula and simplifying:

$$\begin{aligned} \gamma &= -\arg(-w') = -\arg(ic - iw) \\ &= -\arg\left(-\frac{|w|^2}{2\Im w} - iw\right) = -\arg(\Im^2 w - \Re^2 w - 2i\Re w\Im w) \\ &= -\arg(-w^2) = -(\arg(w^2) - \pi) = \pi - 2\arg(w). \end{aligned}$$

Therefore

$$|\mathbf{A}| = \frac{1}{2}(\pi - \alpha - \beta - \gamma) = \arg w = \arg(\|\zeta\|^2 + iw).$$

The integral of equation (6) has negative sign for  $\arg w > 0$  and positive sign for  $\arg w < 0$ . Putting all this together it follows that  $\mathbf{A} = -\arg w = \arg \bar{w} = \arg(\|\zeta\|^2 + iw)$ , as claimed.  $\square$

**2.5. Angles and reflections in  $\mathbf{C}^2$ .** A few elementary results about the geometry of  $\mathbf{C}^2$  are needed and are included here.

The angle between the complex lines  $\text{Span}_{\mathbf{C}} x$  and  $\text{Span}_{\mathbf{C}} y$  in  $\mathbf{C}^n$  is given by:

$$(7) \quad \cos^2 \theta = \frac{\langle\langle x, y \rangle\rangle \langle\langle y, x \rangle\rangle}{\langle\langle x, x \rangle\rangle \langle\langle y, y \rangle\rangle}.$$

**Lemma 2.5.** *Let  $q = (q_1, q_2) \in \mathbf{C}^2$ . The complex reflection in  $\mathbf{C}^2$  about the complex line  $\text{Span}_{\mathbf{C}}(q)$  is given by:*

$$(8) \quad R = \begin{pmatrix} \cos 2\theta & e^{i(\alpha-\beta)} \sin 2\theta \\ e^{i(\beta-\alpha)} \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

where  $\alpha = \arg q_1$ ,  $\beta = \arg q_2$  and  $\theta$  is the angle between  $\text{Span}_{\mathbf{C}} q$  and  $\text{Span}_{\mathbf{C}}(e_1 = (1, 0))$ .

*Proof.* Without losing generality assume that  $\|q\| = 1$ , a simple computation shows that:

$$(9) \quad R = \begin{pmatrix} |q_1|^2 - |q_2|^2 & 2q_1\bar{q}_2 \\ 2\bar{q}_1q_2 & |q_2|^2 - |q_1|^2 \end{pmatrix}.$$

(It is easy to verify that  $R^2 = 1$  and  $Rq = q$ .) Replacing  $x = e_1$  and  $y = q$  in equation (7) we obtain  $\cos^2 \theta = |q_1|^2$ . Then  $\sin^2 \theta = |q_2|^2$ . Since  $q_1 = |q_1|e^{i\alpha}$  and  $q_2 = |q_2|e^{i\beta}$ , the lemma follows.  $\square$

**Lemma 2.6.** *Let  $p, r \in \mathbf{C}^2$  be two non-zero vectors that are not  $\mathbf{R}$ -collinear. Define the real two-plane  $V(p, r) = \text{Span}_{\mathbf{R}}\{p, r\}$ . Denote by  $\theta$  the angle between the complex lines  $\text{Span}_{\mathbf{C}} p$  and  $\text{Span}_{\mathbf{C}} r$ , and let  $\alpha = \arg \langle p, r \rangle$ . Then the angle of holomorphy  $\varphi$  of  $V(p, r)$  is given by the formula:*

$$(10) \quad \cos \varphi = \frac{|\sin \alpha|}{\sqrt{\sin^2 \alpha + \tan^2 \theta}}.$$

*Proof.* We can take  $p$  and  $r$  to be of unit length. Let start by considering the special case where  $p = e_1 = (1, 0)$  and  $r = (r_1, r_2)$ , and write  $V(e_1, r) = V_r$ . Replace  $r$  by the vector  $y = \tilde{y}/\|\tilde{y}\|$ , where  $\tilde{y} = r - (e_1|r)e_1$ . Take two unit vectors  $u = u_1 e_1 + u_2 y \in V_r$  and  $v = v_1 i e_1 + v_2 i y \in V_r$ , the angle of holomorphy  $\varphi$  of the plane  $V_r$  is given by:

$$\begin{aligned} \cos \varphi &= \sup_{u, v} |(u|v)| = \sup_{u, v} [|\Re \langle u_1 e_1, v_2 i y \rangle + \Re \langle v_1 y, u_2 i e_1 \rangle|] \\ &= \sup_{u, v} \left| \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \Re \langle e_1, i y \rangle \right|. \end{aligned}$$

But also  $u_1^2 + u_2^2 = v_1^2 + v_2^2 = 1$ , then the maximum value of this determinant is one. Furthermore  $\|\tilde{y}\| = \sqrt{1 - (\Re r_1)^2} = \sqrt{(\Im r_1)^2 + |r_2|^2}$ , thus

$$\cos \varphi = \frac{|\Im \langle e_1, \tilde{y} \rangle|}{\sqrt{(\Im r_1)^2 + |r_2|^2}} = \frac{|\Im r_1|}{\sqrt{(\Im r_1)^2 + |r_2|^2}}.$$

Finally replace in this last equation  $r_1 = |r_1|e^{i\alpha}$  and remember that  $|r_1| = \cos \theta$  and  $|r_2| = \sin \theta$ , where  $\theta$  is the angle between the complex lines  $\text{Span}_{\mathbf{C}} e_1$  and  $\text{Span}_{\mathbf{C}} r$  defined by (7). Therefore we obtain

$$(11) \quad \cos \varphi = \frac{|\Im e^{i\alpha}|}{\sqrt{(\Im e^{i\alpha})^2 + |r_2|^2/|r_1|^2}} = \frac{|\sin \alpha|}{\sqrt{\sin^2 \alpha + \tan^2 \theta}}.$$

For the general case  $p = (p_1, p_2)$ , the matrix,

$$P = \begin{pmatrix} \bar{p}_1 & \bar{p}_2 \\ -p_2 & p_1 \end{pmatrix}$$

belongs to  $U(2)$  and takes  $p$  to  $e_1$ . Therefore the angle of holomorphy of  $V(p, r)$  equals that of  $V_{Pr}$ , and the angle  $\theta$  between  $\text{Span}_{\mathbf{C}} p$  and  $\text{Span}_{\mathbf{C}} r$  equals the angle between  $\text{Span}_{\mathbf{C}} e_1$  and  $\text{Span}_{\mathbf{C}} Pr$ . The first coordinate of the vector  $Pr$  equals  $\bar{p}_1 r_1 + \bar{p}_2 r_2 = \langle r, p \rangle$ . Then, by equation (11), the lemma follows.  $\square$

**2.6. Triangle groups.** For any three integers  $p, q, r \geq 2$  denote by  $\Delta = \Delta_{(p,q,r)}$  the abstract triangle group with presentation

$$\Delta = \langle a, b, c \mid a^2, b^2, c^2, (ab)^p, (bc)^q, (ca)^r \rangle.$$

Let  $\Delta^+ = \Delta^+_{(p,q,r)}$  denote the index two subgroup generated by the *orientation preserving* elements:  $\alpha = ab$ ,  $\beta = bc$ , and  $\gamma = ca$ . The presentation of this group with these generators is:

$$\Delta^+ = \langle \alpha, \beta, \gamma \mid \alpha^p, \beta^q, \gamma^r, \alpha\beta\gamma \rangle.$$

Abstract triangle groups  $\Delta_{(p,q,r)}$  are of Euclidean, hyperbolic or spherical type depending on whether the quantity  $1/p + 1/q + 1/r$  is equal to, less than or bigger than one, respectively [8]. Euclidean and hyperbolic triangle groups are infinite, whereas spherical groups are finite. Each triangle group admits a standard representation as a group of reflections in the Euclidean plane, the hyperbolic plane and the sphere, respectively. Any group with presentation  $\Delta_{(p,2,2)}$ ,  $p \geq 2$  is spherical. The only triangle Euclidean groups are  $\Delta_{(2,4,4)}$ ,  $\Delta_{(2,3,6)}$  and  $\Delta_{(3,3,3)}$ .

The following theorem and lemma are elementary and the reader is referred to [5] for proofs. The conclusion of the theorem also holds for  $\Delta^+$ .

**Theorem 2.7.** *Let  $\Delta$  be a triangle group of Euclidean or hyperbolic type. Then the element  $\tau = \alpha\gamma\beta = (abc)^2$  has infinite order in  $\Delta$ .*

**Lemma 2.8.** *Let  $\Delta = \Delta_{(p,q,r)}$  be an abstract Euclidean group. Let  $\Delta \rightarrow \text{Isom}(\mathbf{R}^2)$  be a representation of  $\Delta$  such that,*

1. *each generator of  $\Delta$  ( $a$ ,  $b$  and  $c$ ) is identified with a reflection  $s_a$  (respectively  $s_b$  and  $s_c$ ) on a line  $L_a$  (respectively  $L_b$  and  $L_c$ ),*
2. *the lines  $L_a$ ,  $L_b$  and  $L_c$  are not pairwise parallel,*
3. *the products of generators are nontrivial and have orders exactly  $p$ ,  $q$  and  $r$  for  $r_\alpha = s_a s_b$ ,  $r_\beta = s_b s_c$  and  $r_\gamma = s_c s_a$ , respectively.*

*Then the triangle formed by the lines  $L_a$ ,  $L_b$  and  $L_c$  has interior angles  $\pi/p$ ,  $\pi/q$  and  $\pi/r$  at the points  $A = L_a \cap L_b$ ,  $B = L_b \cap L_c$  and  $C = L_c \cap L_a$ . And the rotations  $r_\alpha$ ,  $r_\beta$  and  $r_\gamma$  are all in the same direction with angles  $2\pi/p$ ,  $2\pi/q$  and  $2\pi/r$ , respectively.*

**2.7. Quaternions and Unitary Automorphisms.** The main reference for this section is the book [7]. Define the non commutative field of *quaternions* ( $\mathbb{H}$ ) by

$$\mathbb{H} = \{z + wj \mid z, w \in \mathbf{C}, j^2 = -1, xj = j\bar{x}, \forall x \in \mathbf{C}\}.$$

The *conjugate*  $\bar{x}$  of  $x = z + wj \in \mathbb{H}$  is defined to be  $\bar{x} = \bar{z} - wj$ . And the norm is  $\|x\|^2 = \bar{x}x = |z|^2 + |w|^2$ . The *unit quaternions*, denoted by  $S^1\mathbb{H}$ , are the elements of  $\mathbb{H}$  of norm one. Since  $\|xy\| = \|x\| \cdot \|y\|$ , the unit quaternions form a subgroup of  $\mathbb{H}$ .

Identify  $\mathbf{C}^2$  with  $\mathbb{H}$  by  $(z, w) \leftrightarrow z + wj$ . The group of unit quaternions acting on  $\mathbf{C}^2$  by right multiplication is isomorphic to  $SU(2)$  via the map:

$$\kappa = \alpha + \beta j \mapsto K = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix},$$

for  $K \in SU(2)$  and  $\kappa \in S^1\mathbb{H}$ . Maximal abelian subgroups of  $SU(2)$  are tori. This is not true in general for general compact Lie groups (e.g. the maximal abelian subgroup of  $SO(3)$  is  $\mathbf{Z}/2\mathbf{Z} \times \mathbf{T}$ , where  $\mathbf{T}$  is a maximal torus). A maximal torus  $\mathbf{T}$  for  $SU(2)$  can be identified in  $S^1\mathbb{H}$  with the set of unit complex numbers  $S^1$ .

The following is a standard fact of the Weyl group of  $SU(2)$ ; we will use this result often.

**Lemma 2.9.** *Let  $\mathbf{T} \subset SU(2)$  be a maximal torus. If  $g$  is an element of  $SU(2)$  such that  $g^{-1}\mathbf{T}g \subset \mathbf{T}$  and  $g \notin \mathbf{T}$  then,  $g^{-1}tg = t^{-1}$  for all  $t \in \mathbf{T}$ .*

### 3. REPRESENTATIONS OF TRIANGLE GROUPS IN $\text{Isom}(\mathcal{H}_5)$

**3.1. Explicit description of the Group in  $\mathcal{H}_5$ .** Any vertical 2-chain  $P$  is the inverse image under the vertical projection of a complex line  $l \subset \mathbf{C}^2$ , that is  $P = \Pi_V^{-1}(l)$ . Denote by  $s$  the Heisenberg reflection around  $P$ , and let  $T$  be a Heisenberg translation so that  $T(P)$  contains the origin. If  $\rho$  is the Heisenberg reflection around  $T(P)$ , then

$$s = T^{-1} \circ \rho \circ T.$$

The reflection  $\rho$  is defined by  $\rho(\zeta, v) = (R\zeta, v)$  where  $R \in U(2)$  and  $R^2 = I$ . The matrix  $R$  defines a complex reflection on  $\mathbf{C}^2$  that fixes the complex line  $l = \Pi_V(T(P))$ . We will call  $R$  the *linear part* of  $s$ .

We need to describe the group  $G$ , which is the group generated by the reflections on the three 2-chains  $P_1$ ,  $P_2$  and  $P_3$ .

**Lemma 3.1.** *Let  $\mathcal{C} = \{Q_1, Q_2, Q_3, Q_4\}$  be a set of four different points in  $\partial\mathbf{H}_{\mathbf{C}}^3$ . Then there is a holomorphic isometry of  $\mathbf{H}_{\mathbf{C}}^3$  that maps the set  $\mathcal{C}$  to the set  $\mathcal{P} = \{p_\infty, p_0, p_1, p_q\}$ , so that the horospherical coordinates, centered at  $p_\infty$ , of the points of the set  $\mathcal{P}$  are:*

$$\begin{aligned} p_\infty &= \infty, & p_0 &= ((0, 0), 0) \\ p_1 &= ((1, 0), v_1), & p_q &= (q, v_2) = ((q_1, q_2), v_2). \end{aligned}$$

With  $q = (q_1, q_2) \in \mathbf{C} \times \mathbf{R}$  and  $v_1, v_2 \in \mathbf{R}$ . In particular, the infinite one chains joining  $p_\infty$  with  $p_0$ ,  $p_1$  and  $p_q$  are determined by  $\Pi_V(p_0)$ ,  $\Pi_V(p_1)$  and  $\Pi_V(p_q)$ .

The proof of the first part of this lemma is elementary and it consists on applying the full set of holomorphic isometries of  $\mathbf{H}_{\mathbf{C}}^3$  to the points in  $\mathcal{C}$  (for details see [5]). The claims about the two-chains follow from the description of infinite two chains given above.

From now on we will normalize the points of  $\mathcal{C}$  according to Lemma 3.1. The associated chains will be denoted by  $P_1$  for the vertical chain through  $p_0$  and  $p_1$ ,  $P_2$  for the vertical chain through  $p_0$  and  $p_q$  and  $P_3$  for the vertical chain through  $p_1$  and  $p_q$ . The lines in  $\mathbf{C}^2$  corresponding to these chains are  $l_1 = \text{Span}_{\mathbf{C}} e_1$ ,  $l_2 = \text{Span}_{\mathbf{C}} q$  and  $l_3 = e_1 + \text{Span}_{\mathbf{C}}(q - e_1)$ , respectively.

Remember that the group  $G$  is the group generated by reflections around the chains  $P_1$ ,  $P_2$  and  $P_3$ . By lemma 2.5 the rotational parts  $R_i$  of the reflections on  $\mathbf{C}^2$  about the lines  $l_i$  are:

**Lemma 3.2.**

$$(12) \quad R_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(13) \quad R_2 = \begin{pmatrix} \cos 2\theta & e^{i\alpha} \sin 2\theta \\ e^{-i\alpha} \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$$(14) \quad R_3 = \begin{pmatrix} \cos 2\psi & e^{i\beta} \sin 2\psi \\ e^{-i\beta} \sin 2\psi & -\cos 2\psi \end{pmatrix}$$

where  $\theta$  is the angle between  $\text{Span}_{\mathbf{C}}(q)$  and  $\text{Span}_{\mathbf{C}}(e_1)$ ,  $\psi$  is the angle between  $\text{Span}_{\mathbf{C}}(q - e_1)$  and  $\text{Span}_{\mathbf{C}} e_1$ ,  $\alpha = \arg\langle e_1, q \rangle$  and  $\beta = \arg\langle e_1, q - e_1 \rangle$ .

**Proposition 3.3.** *With the normalization of Lemma 3.1, the group  $G$  is generated by the following Heisenberg-reflections:*

$$(15) \quad s_1(\zeta, v) = (R_1\zeta, v)$$

$$(16) \quad s_2(\zeta, v) = (R_2\zeta, v)$$

$$(17) \quad s_3(\zeta, v) = (R_3 \circ T_{(w,0)})(\zeta, v)$$

$$\text{where } w = R_3 e_1 - e_1 = \begin{pmatrix} \cos 2\psi - 1 \\ e^{-i\beta} \sin 2\psi \end{pmatrix}.$$

*Proof.* The only thing left to do is the derivation of the formula for  $s_3$ :

$$\begin{aligned} s_3 &= T_{(e_1,0)} \circ R_3 \circ T_{(-e_1,0)} = R_3 \circ T_{(R_3 e_1,0)} \circ T_{(-e_1,0)} \\ &= R_3 \circ T_{(R_3 e_1 - e_1, -2\Im\langle R_3 e_1, e_1 \rangle)}. \end{aligned}$$

Using equation (14) it is easy to check that  $\Im\langle R_3 e_1, e_1 \rangle = 0$ .  $\square$

**3.2. Faithfulness on Dihedral Groups.** We will prove now that  $G$  is a representation of an abstract triangle group in  $\text{Isom}(\mathcal{H}_5)$ .

**Proposition 3.4.** *If  $G$  is discrete in  $\mathcal{H}_5$ , the subgroups  $D_{i,j} = \langle s_i, s_j \rangle$ ,  $i < j$  are isomorphic to an abstract dihedral group  $\mathcal{D}_{i,j} = \langle \sigma_i, \sigma_j \mid \sigma_i^2 = \sigma_j^2 = (\sigma_i \sigma_j)^p = 1 \rangle$ .*

*Proof.* Observe that  $D_{1,2}$  fixes the origin and that the subgroups  $D_{1,3}$  and  $D_{2,3}$  can be conjugated to groups fixing the origin as well. It is enough then to prove the lemma for  $D_{1,2}$ . From proposition 3.3  $s_1 s_2(\zeta, v) = (R_1 R_2 \zeta, v)$  is elliptic, but  $D_{1,2}$  is discrete, which implies that  $s_1 s_2$  is of finite order  $p$ , and the lemma follows.  $\square$

The following proposition follows as a direct consequence of this lemma.

**Proposition 3.5.** *The group  $G$  is a representation of a triangle group  $\Delta = \Delta_{(p,q,r)}$  into  $\text{Isom}(\mathcal{H}_5)$ . In other words there exists a group homomorphism  $\phi : \Delta \rightarrow \text{Isom}(\mathcal{H}_5)$  such that  $\phi(\Delta) = G$ .*

*Proof.* Define  $\phi(a) = s_1$ ,  $\phi(b) = s_2$  and  $\phi(c) = s_3$ . From the previous lemma  $s_i s_j$  has finite order, say  $(s_1 s_2)^p = (s_1 s_3)^q = (s_2 s_3)^r = 1$ . This implies that  $\phi$  extends to a homomorphism of the the group  $\Delta$  to  $G$ .  $\square$

Following the notation of proposition 3.4, a representation  $\phi : \Delta \rightarrow G$  is said to be *faithful on dihedral subgroups* if  $\phi|_{\mathcal{D}_{i,j}}$  is injective for all  $i \neq j$ . We say that  $\phi$  is *discrete on dihedral subgroups* if  $\phi|_{\mathcal{D}_{i,j}}$  is injective and discrete, for all  $i \neq j$ . Note that if  $\phi : \Delta \rightarrow \text{Isom}(\mathcal{H}_5)$  is faithful on dihedral subgroup, the representation  $\psi : \Delta \rightarrow \text{Aut}(\mathbf{C}^2)$  defined by  $\psi = \Pi_V \circ \phi$  is also faithful on dihedral subgroups.

#### 4. FINITE GROUPS

We start by analyzing representations of spherical triangle groups in  $\mathcal{H}_5$ . In this section we shall show that a faithful and discrete representation of a spherical group is necessarily trivial.

**Proposition 4.1.** *If the group  $G = \langle \rho_1, \rho_2, \rho_3 \rangle$  has a fixed point inside  $\mathbf{H}_{\mathbf{C}}^3$ , then  $K \cong \mathbf{Z}/2\mathbf{Z}$ .*

*Proof.* The group  $G$  is a subgroup of  $\mathcal{H}_5 \rtimes U(2)$  which is a subgroup of the stabilizer of  $p_\infty$ . The group  $G$  stabilizes also the horospheres centered at  $p_\infty$ . Suppose that the fixed point has horospherical coordinates  $p = (\zeta_0, v_0, u_0) \in \mathbf{H}_{\mathbf{C}}^3$ . Its image under geodesic projection

$$p_g = \Pi_g(p) = (\zeta_0, v_0)$$

must also be fixed by  $K|_{\mathcal{H}_5} = G$ .

Then the point  $p_g$  is fixed by  $G = \langle s_1, s_2, s_3 \rangle$  and in particular by  $s_1, s_2$  and  $s_3$ . The fixed point set of  $s_i$  in  $\mathcal{H}_5$  is equal to

$$\text{Fix}(s_i) = \Pi_V^{-1}(l_i).$$

Therefore  $\Pi_V(p_g) \in \cap_{i=1}^3 l_i$ . The intersection of the three lines  $\{l_i\}_{i=1}^3$  must be nonempty, and this implies that  $q \in \text{Span}_{\mathbf{C}}\{e_1\} = l_1$ . Therefore all three lines are the same. This means that all the points of  $\mathcal{C}$  will be contained in the infinite 2-chain defined by  $\Pi_V^{-1}(l_1)$ , and the proposition follows.  $\square$

**Corollary 4.2.** *Let  $\phi : \Delta \rightarrow G$  be the representation described above. If the abstract group  $\Delta$  is spherical,  $K \cong \mathbf{Z}/2\mathbf{Z}$ .*

*Proof.* Regard  $G$  as a subgroup of  $PU(n, 1)$ . Suppose  $\Delta$  is spherical, therefore it is finite, and the group  $G = \phi(\Delta)$  is finite too. Since  $G$  acts in  $\mathbf{H}_{\mathbf{C}}^3$  by isometries and  $\mathbf{H}_{\mathbf{C}}^3$  has negative curvature  $G$  must have a fixed point in  $\mathbf{H}_{\mathbf{C}}^3$ . By the previous proposition this implies that  $G \cong \mathbf{Z}/2\mathbf{Z}$ . This can only happen if all the vertices of the tetrahedron belong to the same two chain. But this implies that  $K = G$ .  $\square$

**Definition 4.3.** *We will say that a representation  $\phi : \Delta \rightarrow \text{Isom}(H)$  of an abstract group  $\Delta$  into the isometry group of a Lie group  $H$  is almost trivial if  $\phi(\Delta) \cong \mathbf{Z}/2\mathbf{Z}$ . In that case we will say too that  $\phi(\Delta)$  is almost trivial.*

To summarize the results of this section we have:

**Theorem 4.4.** *If the group  $G$  generated by the reflections on three vertical 2-chains is discrete in  $\text{Isom}(\mathcal{H}_5)$  then  $G$  is a representation of an abstract triangle group  $\Delta$ . If the group  $G$  is finite then  $G$  is almost trivial. In particular, this implies that nontrivial representations of spherical triangle groups are always almost trivial.*

## 5. DISCRETE REPRESENTATIONS OF TRIANGLE GROUPS.

The main result of this section is that the representation

$$\phi : \Delta \rightarrow \text{Isom}(\mathcal{H}_5)$$

is real and faithful on dihedral groups iff  $\Delta$  is of Euclidean type. In this case  $\phi$  is also faithful. We also show that representations faithful on dihedral groups of hyperbolic triangle groups in  $\mathcal{H}_5$  are possible. These representations are not longer faithful. We present an example of a representation of the group  $\Delta_{(4,4,4)}$  in  $\text{Isom}(\mathcal{H}_5)$ . This example will be used to complete the proof of the main theorem of this section. A complete classification of these representations can be found in [6]. The methods used in that paper are different than those used here to find the example.

In order to understand under which conditions the group  $G$  is discrete, we need to concentrate on a special subgroup of  $G$ . It was proven in theorem 2.7 for a Euclidean or hyperbolic triangle group that the element  $\tau = (abc)^2$  has infinite order. Its image under  $\phi$  is the isometry defined by  $t = \phi(\tau) = (\sigma_1\sigma_2\sigma_3)^2$ . Assume

that the representation  $\phi$  maps  $a$  to  $s_1$ ,  $b$  to  $s_2$  and  $c$  to  $s_3$ . Define the group  $N(t)$  to be the normal closure in  $G$  of the cyclic group generated by  $t$ . We shall study this subgroup of  $G$ . To that end we need first to explicitly compute  $t$ .

The element  $t$  is the square of the transformation  $\sigma_1\sigma_2\sigma_3 = R \circ T_{(w,0)}$ , where —with the notation of proposition 3.3—  $w = R_3e_1 - e_1$  and  $R = R_1R_2R_3$ . By equations (12), (13) and (14),  $R$  equals

$$(18) \quad R = \begin{pmatrix} u & v \\ \bar{v} & -\bar{u} \end{pmatrix}$$

with

$$\begin{aligned} u &= \cos 2\theta \cos 2\psi + e^{i(\alpha-\beta)} \sin 2\theta \sin 2\psi \\ v &= e^{i\beta} \cos 2\theta \sin 2\psi - e^{i\alpha} \sin 2\theta \cos 2\psi \end{aligned}$$

In particular the determinant of  $R$  is  $-1$  and its characteristic polynomial is:

$$(19) \quad \rho(\lambda) = \lambda^2 - \operatorname{tr} R\lambda - 1,$$

$$(20) \quad \operatorname{tr} R = 2i \Im u = 2i \sin 2\theta \sin 2\psi \sin(\alpha - \beta).$$

Furthermore:

$$(21) \quad t = R \circ T_{(w,0)} \circ R \circ T_{(w,0)} = R^2 \circ T_{(R^{-1}w+w, 2\Im \langle R^{-1}w, w \rangle)}$$

The linear part of the transformation  $t$  is equal to  $R^2$ . The following lemma shows the importance of the study of the matrix  $R$  in this analysis.

**Lemma 5.1.** *Let  $\phi : \Delta \rightarrow G$  be a representation of a triangle group as described in proposition 3.3, and let  $R$  be the linear part of the product  $\sigma_1\sigma_2\sigma_3$ . Assume that the representation is not almost trivial, then the following statements are equivalent:*

- (i)  $R$  has an eigenvalue  $-1$ .
- (ii)  $R$  has order two.
- (iii) The angle of holomorphy of the real 2-plane  $V$  of  $\mathbf{C}^2$  that contains the set of points  $\{p_{12}, p_{23}, p_{31}\} = \Pi_V(\mathcal{L})$  is  $\pi/2$ .
- (iv) The representation  $\phi : \Delta \rightarrow G$  is real.

*Proof.* Since  $\det R = -1$  and  $R \in U(2)$  it is clear that (i) and (ii) are equivalent. Also by the definition of a real representation (iii) and (iv) are equivalent.

We need to prove that (ii) and (iii) are equivalent. From equations 19 and 20 one can see that (ii) is equivalent to:

$$\sin 2\theta \sin 2\psi \sin(\alpha - \beta) = 0$$

Where  $\theta$ ,  $\psi$ ,  $\alpha$  and  $\beta$  are as defined by lemma 3.2. The values of these angle range from 0 to  $\pi/2$  therefore the solutions of this equation are  $\theta = 0$  or  $\pi/2$ ,  $\psi = 0$  or  $\pi/2$  or  $\alpha = \beta$ .

If  $\theta = 0$  or  $\psi = 0$ , the three lines  $l_1$ ,  $l_2$  and  $l_3$  coincide and the group  $G$  is almost trivial.

Applying lemma 2.6 with  $p = e_1$ ,  $r = q$  and  $\theta = \pi/2$ , the angle of holomorphy  $\varphi$  is equal to  $\pi/2$ . Analogously, with  $p = e_1$  and  $r = q - e_1$  and  $\psi = \pi/2$ , the angle of holomorphy is again equal to  $\pi/2$ .

Finally if  $\alpha = \beta$  then  $\arg q_1 = \arg(q_1 - 1)$  which can only happen if  $q_1 \in \mathbf{R}$ , therefore  $\alpha = 0$ , and equation (10) shows that  $\varphi = \pi/2$ .  $\square$

The matrix  $R$  belongs to  $U(2)$ , and its order gives information about the representation  $\phi : \Delta \rightarrow G$ . Moreover  $\det R^2 = 1$ , therefore the linear part of  $t$  belongs to  $SU(2)$ .

Let us assume first that the order of  $R$  is finite; the infinite order case will be studied later.

**Lemma 5.2.** *Suppose that  $\phi : \Delta \rightarrow G$  is a representation of a triangle group  $\Delta$  that is not almost trivial. Suppose that  $R = R_1 R_2 R_3$  is a matrix of finite order. Then, the order of  $R$  is an even integer  $2n$ . If  $n = 1$  the representation  $\phi$  is real and  $t$  is a non-zero Heisenberg translation that leaves the real plane  $V$  that contains  $\Pi_V(\mathcal{L})$  invariant. If  $n > 1$ , then  $\phi$  is not real and  $t^n$  is a vertical translation.*

*Proof.* The order of  $R$  is even because  $\det R = -1$ . If  $n = 1$  the only parts of this lemma that are not a restatement of lemma 5.1 are the nontriviality of  $t$  and that  $t$  leaves  $V$  invariant. The isometry  $t$  is the image under  $\phi$  of  $\tau = (abc)^2 \in \Delta$ . The orientation preserving group  $\Delta^+$  of  $\Delta$  (section 2.6) maps to an index 2 subgroup  $G^+$  of  $G$ . Suppose that  $t = \phi(\tau) = 1$ . In  $\Delta^+$  the element  $\tau$  is equal to  $\alpha\beta\gamma$  and the group  $\Delta^+/N(\tau)$  has presentation:

$$\Delta^+/N(\tau) \cong \langle \beta, \gamma \mid (\beta\gamma)^p, \beta^q, \gamma^r, \beta^{-1}\gamma^{-1}\beta\gamma \rangle$$

which is a subgroup of the finite group  $\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/q\mathbf{Z}$ , and would map surjectively onto  $G^+$  via the map induced by  $\phi$ . Therefore  $G^+$  is finite. By theorem 4.4  $G$  must be almost trivial, a contradiction. We have then that  $t$  is nontrivial.

We need to prove that  $t$  leaves  $V = \text{Span}_{\mathbf{R}}(\Pi_V(\mathcal{L}))$  invariant. Up to a change of coordinates assume that the points of  $\Pi_V(\mathcal{L})$  are  $o$ ,  $e_1$  and  $q = (q_1, q_2)$ , with  $q_1 \in \mathbf{C}$  and  $q_2 \in \mathbf{R}$ . With this notation  $V = \text{Span}_{\mathbf{R}}(e_1, q)$ . From the proof of lemma 5.1 we have that  $R$  has order two if  $\sin(\alpha - \beta) \sin(2\psi) \sin(2\theta) = 0$ , where  $\alpha$ ,  $\beta$ ,  $\theta$  and  $\psi$  are as defined in lemma 3.2. As it was discussed in that proof  $\theta = 0$  (or  $\psi = 0$ ) implies that  $G \cong \mathbf{Z}/2\mathbf{Z}$ . Therefore we analyze the other solutions. If  $\theta = \angle(\mathbf{C} \cdot e_1, \mathbf{C} \cdot q) = \pi/2$  or if  $\psi = \angle(\mathbf{C} \cdot e_1, \mathbf{C} \cdot q - e_1) = \pi/2$  from equation (7) it is clear that  $q \in \mathbf{R}^2$ . Similarly if  $\alpha = \arg(q_1) = 0$  (equivalently  $\beta = \arg(q_1 - 1) = 0$ )  $q_1 \in \mathbf{R}$ , so  $q \in \mathbf{R}^2$ . Finally, if  $q \in \mathbf{R}^2$  then  $V \cong \mathbf{R}^2$  and  $t$  is a real translation that fixes  $V$  as claimed.

Suppose  $n > 1$  by equation (21) the element  $t$  equals

$$R^2 \circ T_{(R^{-1}w+w, 2\Im\langle R^{-1}w, w \rangle)}.$$

In fact  $R^{-1}w + w \neq 0$ . If not either  $w \neq 0$  and  $Rw = -w$  or  $w = 0$ . The former is impossible by lemma 5.1, the latter by observing from the explicit formula for  $w$  (lemma 3.2) that  $w = 0$  would imply that  $\psi = 0$ , and hence  $G \cong \mathbf{Z}/2\mathbf{Z}$ . (See proof of lemma 5.1.) For simplicity let  $\zeta = R^{-1}w + w$  and  $A = R^2$ . Call  $t_c$  the map in  $\mathbf{C}^2$  defined by  $\Pi_V \circ t = t_c \circ \Pi_V$ . Then  $t_c(z) = Az + \zeta$ , and

$$t_c^n(z) = z + \sum_{j=0}^{n-1} A^j \zeta.$$

But  $(A - I) \sum_{j=0}^{n-1} A^j \zeta = 0$ . So either  $\sum_{j=0}^{n-1} A^j \zeta = 0$  or 1 is an eigenvalue for  $A$ . If 1 is an eigenvalue for  $A \in SU(2)$ ,  $A$  must be the identity matrix, and  $n = 1$ . Therefore  $\sum_{j=0}^{n-1} A^j \zeta = 0$ ,  $t_c^n = I$ , and  $t^n$  is a vertical translation, as claimed.  $\square$

As a direct consequence of this lemma we have:

**Corollary 5.3.** *Suppose  $\phi : \Delta \rightarrow \text{Isom}(\mathcal{H}_5)$  is discrete and faithful on dihedral subgroups and  $\Delta$  is hyperbolic, then the representation  $\phi$  is not real.*

*Proof.* Suppose that  $\phi$  is real, in that case we can restrict  $\phi$  to a totally real subspace of  $\mathbf{C}^2$ . However, it is not possible to find a representation of a hyperbolic triangle group in  $\mathbf{R}^2$  which is faithful on dihedral subgroups.  $\square$

And in fact if  $G$  is discrete and the representation  $\phi$  is real, the group  $\Delta$  must be Euclidean. If  $\phi$  is real, then  $\bar{\phi} = \Pi_V \circ \phi$  leaves a real subspace  $W$  of  $\Pi_V(\mathcal{H}_5) \cong \mathbf{C}^2$  invariant, and since  $\phi$  is discrete, then  $\bar{\phi}$  is a discrete representation of a triangle group in  $W \cong \mathbf{R}^2$ , then  $\Delta$  must be of Euclidean type. The converse is also true but much more difficult to prove, first we prove that the matrix  $R$  defined above has order 2 and invoke lemma 5.1.

From now on unless it is specifically stated we assume that  $\Delta$  is a Euclidean group and that  $\phi : \Delta \rightarrow G \subset \text{Isom}(\mathcal{H}_5)$  is a discrete representation, faithful on dihedral subgroups, that is not almost trivial.

**Theorem 5.4.** *The group  $\Delta$  is Euclidean if and only if the representation  $\phi$  is real.*

In that case by lemma 2.8 the representation  $\phi$  is also faithful, since it is just the extension of a representation of a Euclidean group in  $\mathbf{R}^2$  to the Heisenberg group.

To prove this theorem we will need to show that if  $\Delta$  is Euclidean, the representation  $\phi$  is real. By lemma 5.1 this is equivalent to  $A = I$ . We will show this by using the structure of the group  $\Delta$  in the Euclidean case and by studying the projection of the group  $N(\tau) \subset \Delta^+$  into  $SU(2)$ . We will proceed by contradiction, assuming that  $A \neq I$  and proving that this is impossible if the group  $\Delta$  is Euclidean. (This is not longer true in if  $\Delta$  is hyperbolic.) Two cases arise:  $A$  of infinite order, and  $A$  of finite order. For the case of  $A$  of finite order we need two results that are explained in the following lemma and proposition.

Before we can continue we need to fix some notation. Consider the representation  $\phi^+ : \Delta^+ \rightarrow G^+ \subset \text{Isom}^+(\mathcal{H}_5) \cong \mathcal{H}_5 \rtimes SU(2)$ , of the orientation preserving group  $\Delta^+$ . We will compose this representation with the natural projection  $\pi : \text{Isom}^+(\mathcal{H}_5) \rightarrow SU(2)$  to obtain a representation  $\phi' : \Delta^+ \rightarrow G' = \pi(G^+) \subset SU(2)$ , using the properties of this Lie group most of the results will follow. We fix the notation  $x_\alpha, x_\beta$  and  $x_\gamma$  for the images under  $\phi'$  in  $SU(2)$  of the generators  $\alpha, \beta$  and  $\gamma$  of  $\Delta^+$ , respectively. From lemma 3.2 we have  $x_\alpha = R_1 R_2, x_\beta = R_2 R_3$  and  $x_\gamma = R_3 R_1$ .

**Lemma 5.5.** *Assume that  $x_\alpha$  has order bigger than two, and  $R$  has order different from four. If  $x_\alpha$  commutes with  $R^2$  then  $\phi$  is a real representation.*

*Proof.* We need to prove that  $R$  has order two. Let  $A = R^2$ , Since  $\Delta$  is Euclidean, the group  $N(A) = \pi(N(t))$  is abelian. Then it must be contained in a maximal abelian subgroup of  $SU(2)$ .

Assume that  $A^n = I, n > 1$ . Since  $n > 2, N(A)$  has at least three elements. Let  $\mathbf{T}$  be the maximal torus that contains  $N(A)$ . Since  $g^{-1}N(A)g = N(A)$ , for all  $g \in \pi(G^+)$ , then  $g^{-1}\mathbf{T}g = \mathbf{T}$  for all  $g \in G^+$ . We have that  $x_\gamma x_\beta = x_\alpha^{-1}A \in \mathbf{T}$ . Then either  $x_\gamma \in \mathbf{T}$ , and  $A = x_1 x_\gamma x_\beta = x_\alpha x_\beta x_\gamma = 1$ , or  $x_\gamma \notin \mathbf{T}$ . By lemma 2.9  $x_\gamma^{-1}(x_\gamma x_\beta)x_\gamma = (x_\gamma x_\beta)^{-1}$  which implies that  $x_\beta^2 x_\gamma^2 = 1$ .

Two of the three Euclidean groups contain an element of order two. If  $x_\beta^2 = 1$  or  $x_\gamma^2 = 1$ , this last equation implies that  $x_\gamma^2 = 1$  or  $x_\beta^2 = 1$ , respectively. In that case then the group  $\Delta$  would be type  $(p, 2, 2)$ , which is spherical, and this is

impossible. The other Euclidean group is of type  $(3, 3, 3)$ . Then  $x_\beta^2 = x_\gamma^{-2} = (x_\gamma^2)^2 = x_\gamma^4 = x_\gamma$ . So,  $x_\gamma$  and  $x_\beta$  commute and  $A = I$ .  $\square$

The case of  $R$  of order four remains to be ruled out. In fact we obtain the surprising result:

**Proposition 5.6.** *Suppose that  $R$  has order four, and that  $\phi : \Delta \rightarrow \text{Isom}(\mathcal{H}_5)$  is a representation that is discrete on dihedral groups, then  $\Delta$  is hyperbolic of type  $(4, 4, 4)$  and  $G$  is generated by  $s_a = \phi(a)$ ,  $s_b = \phi(b)$  and  $s_c = \phi(c)$ . If under a change of coordinates we make  $\Pi_V(\mathcal{L}) = \{o, e_1, q\}$  with  $\Re q_2 > 0$ , then  $G$  is generated by the Heisenberg isometries  $s_a(\zeta, v) = R_1(\zeta, v)$ ,  $s_b(\zeta, v) = R_2(\zeta, v)$  and  $s_c(\zeta, v) = (R_3 \circ T_{(w,0)})(\zeta, v)$ , with*

$$(22) \quad \begin{aligned} R_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & R_2 &= \begin{pmatrix} 0 & i\xi \\ -i\bar{\xi} & 0 \end{pmatrix} \\ R_3 &= \begin{pmatrix} 0 & \xi \\ \bar{\xi} & 0 \end{pmatrix} & w &= \begin{pmatrix} -1 \\ \bar{\xi} \end{pmatrix} \end{aligned}$$

where  $\xi$  is a unit complex number. All these representations are conjugate by a Heisenberg rotation. Moreover,  $t^2 = (s_a s_b s_c)^4 = T_{(0, \pm 8)}$  and  $G' = \phi'(\Delta^+)$  ( $\subset SU(2) \cong S^1\mathbb{H}$ ) is a group isomorphic to the integer unit quaternions.

*Proof.* Since  $R$  has order 4 and  $\det R = -1$  its eigenvalues must be both equal to either  $i$  or  $-i$ . From equations (19) and (20) we have that this condition implies that:

$$\sin(2\theta) \sin(2\psi) \sin(\alpha - \beta) = \pm 1$$

(with  $\theta$ ,  $\psi$ ,  $\alpha$  and  $\beta$  as in lemma 3.2). Therefore  $\theta = \psi = \pi/4$  and  $\alpha - \beta = \pm\pi/2$ . Replacing these values in equations (12), (13), and (14) we obtain (22). A computation using equation (21) shows that  $t^2 = T_{(0, \pm 8)}$ . Finally it is easy to check that the representation  $\phi' : \Delta^+ \rightarrow G' \subset SU(2) \cong \mathbb{H}$  maps  $\Delta^+$  onto the integer unit quaternions  $\{\pm i, \pm j, \pm k\}$ , in fact  $\phi'(x_\alpha) = i$ ,  $\phi'(x_\beta) = j$  and  $\phi'(x_\gamma) = k$ .  $\square$

Using the identification given by equations (1) and (2) for the matrices that generate  $G$  in the previous theorem, we obtain:

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \sigma_2 &= \begin{pmatrix} 0 & i\xi & 0 & 0 \\ -i\bar{\xi} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \sigma_3 &= \begin{pmatrix} 0 & \xi & 1 & 1 \\ \bar{\xi} & 0 & -\bar{\xi} & -\bar{\xi} \\ 1 & -\xi & 0 & -1 \\ -1 & \xi & 1 & 0 \end{pmatrix} \end{aligned}$$

Conjugating these matrices by

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{\xi} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

we obtain

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{pmatrix}.$$

Hence  $G = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$  is conjugate to a subgroup of  $PU(3, 1; \mathbf{Z}[i])$ , and therefore is discrete. (We would like to thank J. Parker for calling this fact to our attention.)

We are ready to prove theorem 5.4.

*Proof.* (of Theorem 5.4) It remains to be shown that if  $\Delta$  is Euclidean then  $\phi$  is a real representation, equivalently, by lemma 5.1, that  $A = R^2 = I$ .

*Case I* Assume first that  $A$  has infinite order, call  $N(A)$  the image under  $\bar{\phi}$  of  $N(\tau) \subset \Delta^+$  (see section 2.6 for the definition of  $N(\tau)$ ). Since  $\Delta$  is Euclidean,  $N(\tau)$  is abelian and normal, therefore  $N(A) \subset SU(2)$  is abelian and it is contained in a maximal torus of  $SU(2)$ . From now on we will use the identification  $SU(2) = S^1\mathbb{H}$  and we will use the notation introduced in section 2.7.

Since  $N(A)$  is abelian it is contained in a maximal torus  $\mathbf{T} \cong S^1$ . The group  $G' \subset SU(2)$  normalizes  $N(A)$ , in particular  $x_\alpha^{-1}N(A)x_\alpha = N(A)$ . Since  $N(A)$  is infinite, by continuity  $x_\alpha^{-1}\mathbf{T}x_\alpha = \mathbf{T}$ . Without loss of generality assume that  $x_\alpha$  has order bigger than two, then if  $x_\alpha \in \mathbf{T}$ , by lemma 5.5  $A = I$ , and therefore  $x_\alpha \notin \mathbf{T}$ . By lemma 2.9  $x_\alpha^{-2}tx_\alpha^2 = t$ , so  $x_\alpha^2 \in \mathbf{T} \cong S^1$ . The only elements not in  $S^1$  that normalize it and satisfy this property have the form  $bj$ , with  $b \in S^1$ . But then  $x_\alpha^4 = 1$ , and since  $\Delta$  is Euclidean  $\Delta = \Delta_{(2,4,4)}$ , and either  $x_\beta$  or  $x_\gamma$  must have order 2. The only element of order two in  $S^1\mathbb{H}$  is  $-1$ , which is central, therefore  $A = x_\alpha x_\gamma x_\beta = x_\alpha x_\beta x_\gamma = 1$ .

*Case II* Suppose  $A$  has finite order  $n$ . By proposition 5.6  $n > 2$ . Therefore  $N(A)$  has at least three elements and  $g^{-1}\mathbf{T}g = \mathbf{T}$  for all  $g \in G'$ . If either  $x_\alpha$ ,  $x_\beta$  or  $x_\gamma$  have order two a computation similar to the one done in *Case I* shows that  $A = I$ . This takes care of the triangle groups  $\Delta_{(2,4,4)}$  and  $\Delta_{(2,3,6)}$ . The only Euclidean triangle group that remains is  $\Delta_{(3,3,3)}$ . By lemma 5.5  $x_\alpha \notin \mathbf{T}$ , and by lemma 2.9  $x_\alpha^{-2}tx_\alpha^2 = t$ , for all  $t \in \mathbf{T}$ , but then  $x_\alpha = bj$ ,  $b \in S^1$ , and  $x_\alpha^4 = 1$ , which implies that  $x_\alpha = 1$  contradicting proposition 3.4.  $\square$

## 6. TETRAHEDRAL GROUPS

Using the results of the previous section we shall prove the rigidity result that was stated in the introduction (Theorem 1.2). In this section let  $K$  be a complex tetrahedral group associated to the tetrahedron with set of vertices  $\mathcal{C}$ . And for every vertex  $Q_i \in \mathcal{C}$ , let  $G_i$  be the associated subgroup generated by the three generators of  $K$  that fix that vertex.

**Definition 6.1.**  *$K$  is said to be of Euclidean type, if for all the vertices  $Q_i \in \mathcal{C}$ ,  $G_i$  is the representation of a Euclidean triangle group  $\Delta$  in the subgroup of  $PU(n, 1)$  that stabilizes that vertex.*

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\*Here  $S^1$  stands for the unit complex numbers  $S^1 \subset S^1\mathbb{H}$

**Corollary 6.2.** *Let  $\mathcal{C}$ ,  $K$  and  $G_i$  as above. Let  $\mathcal{H}_5^i$  be a Heisenberg space obtained by taking horospherical coordinates with center at  $Q_i$ . Suppose the group  $K$  is of Euclidean type and discrete. Then,*

- i) *the group  $G_i$  is a representation  $\phi_i : \Delta \rightarrow G_i \text{Isom}(\mathcal{H}_5^i)$ , of a fixed Euclidean group  $\Delta = \Delta_{(p,q,r)}$  (for all  $i = 1, \dots, 4$ ).*
- ii) *if  $\pi^i$  represents the vertical projection in  $\mathcal{H}_5^i$ , and  $Q_i^j$  the projection of  $Q_i$  in  $\mathcal{H}_5^j$  (for  $i \neq j$ ), then the triangles with vertices:*

$$\begin{aligned} VT_1 &= \{\pi^1(Q_2^1), \pi^1(Q_3^1), \pi^1(Q_4^1)\} \subset \mathbf{C}^2 \subset \mathcal{H}_5^1 \\ VT_2 &= \{\pi^2(Q_1^2), \pi^2(Q_3^2), \pi^2(Q_4^2)\} \subset \mathbf{C}^2 \subset \mathcal{H}_5^2 \\ VT_3 &= \{\pi^3(Q_1^3), \pi^3(Q_2^3), \pi^3(Q_4^3)\} \subset \mathbf{C}^2 \subset \mathcal{H}_5^3 \\ VT_4 &= \{\pi^4(Q_1^4), \pi^4(Q_2^4), \pi^4(Q_3^4)\} \subset \mathbf{C}^2 \subset \mathcal{H}_5^4. \end{aligned}$$

*are contained in totally real subspaces of  $\mathbf{C}^2$ .*

*Proof.* We prove (i) first. The product of two of these reflections is a rotation  $\nu_{ij}$  of order bigger than one, around the complex line spanned by the vectors  $\mathcal{C} - \{Q_i, Q_j\}$ . Figure 1 shows the configuration of planes and lines. It is easy to check that if  $\nu_{12}^p = \nu_{13}^q = \nu_{23}^r = 1$ , then  $\nu_{34}^p = \nu_{24}^q = \nu_{14}^r = 1$ . Therefore the group  $\Delta = \Delta_{(p,q,r)}$  is the same Euclidean group for all the representations  $\phi_i : \Delta \rightarrow G_i \subset \text{Isom}(\mathcal{H}_5^i)$ .

For the proof of (ii) apply theorem 5.4 to each of the subgroups  $G_i$ .  $\square$

**Proposition 6.3.** *Let  $\mathcal{L} = \{p_0, p_1, p_2\}$  be a set of three different points in  $\mathcal{H}_5$ . Suppose that  $\Pi_V(\mathcal{L})$  is contained in a totally real affine subspace of  $\mathbf{C}^2$ . Let  $c_0, c_1$  and  $c_2$  be the unique vertical one-chains in  $\mathcal{H}_5$  through the points  $p_0, p_1$  and  $p_2$ , respectively. Then, for any point  $q_0 \in c_0$  there are unique points  $q_1 \in c_1$  and  $q_2 \in c_2$  such that the set of points  $\mathcal{Q} = \{q_0, q_1, q_2\}$  is contained in an infinite  $\mathbf{R}^3$ -sphere.*

*Proof.* Moving  $p_0$  by an isometry if necessary, assume without losing generality that  $p_0$  is the origin of  $\mathcal{H}_5$ . Theorem 2.2 states that any infinite  $\mathbf{R}^3$ -sphere through  $q_0$  is an affine subspace of  $\mathcal{H}_5$  that is mapped injectively by  $\Pi_V$  onto a totally real subspace of  $\mathbf{C}^2$ .

Let  $\mathcal{E} = \{E_p\}_{p \in \mathcal{H}_5} \subset T(\mathcal{H}_5)$  be the canonical contact structure of  $\mathcal{H}_5$  induced by the canonical CR structure of  $\partial\mathbf{H}_{\mathbf{C}}^3$ , see 2.2.3.  $\mathbf{R}^3$ -spheres are contact submanifolds of this contact structure. These  $\mathbf{R}^3$ -spheres (of dimension 2) are Lagrange surfaces of  $\mathcal{H}_5$  (of topological dimension 5). Consider the vertical projection  $\Pi_V : \mathcal{H}_5 \rightarrow \mathbf{C}^2$ . Given a totally real subspace  $V \subset \mathbf{C}^2$  and a point  $p \in \mathcal{H}_5$  there is a unique horizontal lifting  $S$  of  $V$  such that  $p \in S$ . This means that

$$T_x(S) \subset E_x$$

for all  $x$  in  $S$ . This horizontal lifting is an  $\mathbf{R}^3$ -sphere.

Let  $V$  be the real subspace of  $\mathbf{C}^2$  that contains  $\Pi_V(\mathcal{L})$ , and let  $S_V^0$  be the lifting of  $V$  to an  $\mathbf{R}^3$ -sphere through  $q_0$ . By hypothesis  $c_1$  and  $c_2$  are vertical chains, then  $\Pi_V(c_i) = p_i$  for  $i = 1, 2$ , therefore the intersection of  $c_i$  with  $S_V^0$  is a unique point, define  $q_1 = c_1 \cap S_V^0$  and  $q_2 = c_2 \cap S_V^0$ .  $\square$

The following geometrical theorem is a consequence of the previous proposition.

**Theorem 6.4.** *Let  $\mathcal{C} = \{p_\infty, p_0, p_1, p_2\}$  be a set of four different points in  $\partial\mathbf{H}_{\mathbf{C}}^3$ . Suppose that the vertical projection from  $p_\infty$  (respectively  $p_0$ ) of the points  $p_0, p_1$*

and  $p_2$  (respectively the points  $p_\infty, p_1$  and  $p_2$ ) is contained in a totally real subspace  $V_0$  (respectively  $V_\infty$ ) of  $\mathbf{C}^2$ . Then

$$\mathbf{A}(p_\infty, p_0, p_1) + \mathbf{A}(p_0, p_\infty, p_2) = 0$$

Before proving this theorem it is necessary to prove some lemmas.

**Lemma 6.5.** *In horospherical coordinates the chain  $c$  corresponding to the unique complex geodesic from the point  $o = (0, 0)$  to the point  $p = (z, u)$  is defined by the equations:*

$$(23) \quad v = -2\Im\langle\zeta, \zeta_0\rangle, \quad \|\zeta - \zeta_0\| = \|\zeta_0\|$$

and  $\zeta$  belongs to the subspace  $V$  of  $\mathcal{H}_5$ , which is defined to be the unique infinite two-chain containing  $o$  and  $p$ . The vector  $\zeta_0$  is defined by:

$$\zeta_0 = \frac{(\|z\|^2 + iu)}{2\|z\|^2} z.$$

Moreover, the chain  $c$  intersects the plane  $v = 0$  only at the points  $o$  and  $r = (2\zeta_0, 0)$ .

*Proof.* The general equation (see (5)) for a chain  $c$  with center  $O_c = (\zeta_0, v_0)$  and radius  $r_0$  is:

$$v = v_0 - 2\Im\langle\zeta, \zeta_0\rangle, \quad \|\zeta - \zeta_0\| = r_0,$$

for  $\zeta \in \mathbf{V}$ . Choose  $r_0 = \|\zeta_0\|$ ,  $\zeta_0 = \frac{(\|z\|^2 + iu)}{2\|z\|^2} z$  and  $v_0 = 0$ . It is easy to verify that the points  $o$  and  $p$  belong to the chain  $c$ . The first part of the proof is obtained by the uniqueness of  $c$ . Solving the equations for the chain given by (23) and  $v = 0$  one obtains that  $\zeta = k\zeta_0$ , with  $k = 0, 2$ . The two intersection points are then,  $o$  and  $r = (2\zeta_0, 0)$ .  $\square$

**Lemma 6.6.** *Any infinite  $\mathbf{R}^3$ -sphere  $S$  of  $\mathcal{H}_{2n-1}$  that contains the origin is an affine subset of  $\mathbf{C}^{n-1}$ , defined in  $\mathcal{H}_{2n-1}$  by  $v = 0$ .*

*Proof.* Theorem 2.2 states that  $S$  is an affine subspace of  $\mathcal{H}_{2n-1}$ , on the other hand  $S$  is a contact surface (section 2.3). At zero the contact plane is defined by the equation  $dv = 0$  (remember that a calibration for the canonical contact structure in  $\mathcal{H}_{2n-1}$  is given by  $\omega = dv + 2\Im\langle d\zeta, \zeta\rangle$ ), and the lemma follows.  $\square$

We can now prove theorem 6.4.

*Proof.* (of Theorem 6.4.) Assume without loss of generality that  $p_0 = ((0, 0), 0)$ , the origin of  $\mathcal{H}_5$ . Let  $\Pi_V^\infty$  and  $\Pi_V^0$  be the vertical projections from  $p_\infty$  and  $p_0$ , respectively. Let  $c_i$  be the chain corresponding to the unique complex geodesic from  $p_i$  to  $p_\infty$ , and  $c_{0i}$  be the chain corresponding to the unique complex geodesic from  $p_i$  to  $p_0$ .

With the notation of the previous proposition choose  $q_0 = p_0$ . The points  $\Pi_V^\infty(\mathcal{L})$  are by hypothesis contained in a totally real subspace  $V_0$  of  $\mathbf{C}^2$ . Then by proposition 6.3 there is a unique  $\mathbf{R}^3$ -sphere,  $S_V^0$ , that contains  $p_0$  and projects to  $V_0$ . Let  $q_1$  and  $q_2$  be the unique points of intersection of  $c_1$  and  $c_2$  with  $S_V^0$ , respectively.

Analogously consider vertical projection from  $p_0$ , in this case

$$\Pi_V^0(\mathcal{C} - \{p_0\}) = \{\Pi_V^0(p_\infty), \Pi_V^0(p_1), \Pi_V^0(p_2)\}$$

is contained in a totally real subspace  $V_\infty$  of  $\mathbf{C}^2$ . By the previous proposition there is a unique  $\mathbf{R}^3$ -sphere ( $S_V^\infty$ ) that contains  $p_\infty$  and projects to  $V_\infty$ . Let  $r_1$  and  $r_2$  be

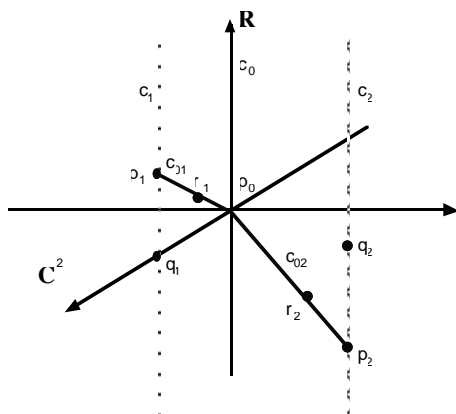


FIGURE 3. Configuration of chains between the points of  $\mathcal{C}$ .

the unique points of intersection of  $c_{01}$  and  $c_{02}$  with  $S_V^\infty$ , respectively. (Compare with figure 3.)

Lemma 6.6 implies that  $S_V^\infty$  and  $S_V^0$  are contained in  $\mathbf{C}^2$  ( $v = 0$ ). In particular this implies that the points  $q_1, q_2, r_1$  and  $r_2$  are also contained in  $\mathbf{C}^2$ . Define the subspaces  $V_i$  of  $\mathcal{H}_5$  by  $V_i = \Pi_V^{-1}(\text{Span}_{\mathbf{C}}\{\Pi_V(p_i)\})$ , for  $i = 1, 2$ . These are infinite two-chains, therefore they bound a complex two-space (and each can be identified with a 3-dimensional Heisenberg space). By hypothesis the points  $p_0, p_\infty, p_i, q_i$  and  $r_i$ , will be contained in  $V_i$ , for  $i = 1, 2$ .

On each subspace the configuration of real and complex subspaces is shown in Figure 4, the dotted lines represent the subspaces  $S_V^0 \cap V_i$  and  $S_V^\infty \cap V_i$ .

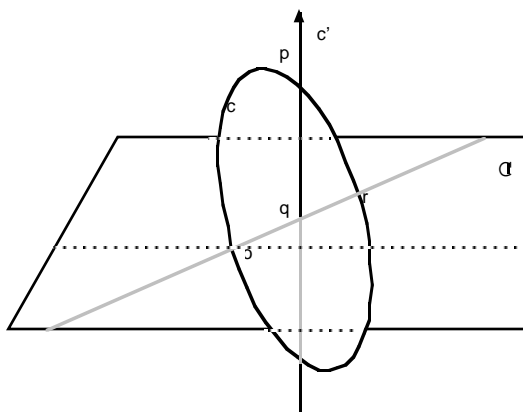


FIGURE 4. Subspace  $V_i$ .

By lemma 6.5 the points  $r_1$  and  $r_2$  are given by:

$$r_1 = \left( \frac{(\|\zeta_1\|^2 + iv_1)}{2\|\zeta_1\|^2} \zeta_1, 0 \right), \quad r_2 = \left( \frac{(\|\zeta_2\|^2 + iv_2)}{2\|\zeta_2\|^2} \zeta_2, 0 \right).$$

The points  $q_1$  and  $q_2$  are  $(\zeta_1, 0)$  and  $(\zeta_2, 0)$ , respectively. Define the planes

$$\begin{aligned} M_1 &= \Pi_V(S_V^0) = \text{Span}_{\mathbf{C}}\{\Pi_V(q_1), \Pi_V(q_2)\} \quad \text{and} \\ M_2 &= \Pi_V(S_V^\infty) = \text{Span}_{\mathbf{C}}\{\Pi_V(r_1), \Pi_V(r_2)\}. \end{aligned}$$

These planes are totally real subspaces of  $\mathbf{C}^2$  and therefore there exists a number  $\xi \in \mathbf{C}$ ,  $|\xi| = 1$  such that  $M_2 = \xi M_1$ . This is possible if and only if

$$\arg(\|\zeta_1\|^2 + iv_1) = \arg(\|\zeta_2\|^2 + iv_2).$$

By lemma 2.4 this is equivalent to  $\mathbf{A}(p_\infty, p_0, p_1) = \mathbf{A}(p_\infty, p_0, p_2)$ .  $\square$

We call a subgroup of  $PO(3, 1)$  real tetrahedral if it is generated by the reflections on the four faces of an ideal tetrahedron in  $\mathbf{H}_{\mathbf{R}}^3$ . Define the real tetrahedral groups  $G_{(p,q,r)}$  by the groups of reflections on a tetrahedron  $\mathcal{T}_{(p,q,r)}$  that has the property that the intersection with every sufficiently small horosphere centered at one of the vertices of the tetrahedron is a triangle of interior angles  $(\pi/p, \pi/q, \pi/r)$ .

**Theorem 6.7.** *Every discrete real tetrahedral subgroup of  $PO(3, 1)$  is conjugate to either  $G_{(3,3,3)}$ ,  $G_{(2,4,4)}$  or  $G_{(2,3,6)}$ .*

Notice that the inclusion  $PO(3, 1) \subset PU(3, 1)$  embeds these groups as examples of complex tetrahedral groups. In other words, the complexifications of the tetrahedrons  $\mathcal{T}_{(p,q,r)} \subset \mathbf{H}_{\mathbf{R}}^3 \subset \mathbf{H}_{\mathbf{C}}^3$  produce examples of discrete complex hyperbolic groups.

We now prove the main theorem:

**Theorem 6.8. (Main Theorem)** *Every discrete complex tetrahedral subgroup of  $PU(3, 1)$  of Euclidean type is the complexification of a discrete real tetrahedral subgroup of  $PO(3, 1)$ .*

*Proof.* Consider the tetrahedron with vertices in  $\mathcal{C} = \{p_\infty, p_0, p_1, p_2\}$  and let

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{A}(p_\infty, p_0, p_1) & \mathbf{A}_2 &= \mathbf{A}(p_0, p_1, p_2) \\ \mathbf{A}_3 &= \mathbf{A}(p_1, p_2, p_\infty) & \mathbf{A}_4 &= \mathbf{A}(p_2, p_\infty, p_0) \end{aligned}$$

be the corresponding Cartan invariants (see section 2.4).

Corollary 6.2 implies that any two different sets of three different points of  $\mathcal{C}$  satisfy the conditions for theorem 6.4. Therefore

$$\begin{aligned} \mathbf{A}_1 - \mathbf{A}_2 &= 0 \\ \mathbf{A}_1 + \mathbf{A}_3 &= 0 \\ \mathbf{A}_1 - \mathbf{A}_4 &= 0. \end{aligned}$$

Adding these three equations one obtains:

$$2\mathbf{A}_1 + \mathbf{A}_1 - \mathbf{A}_2 + \mathbf{A}_3 - \mathbf{A}_4 = 0.$$

By theorem 2.3 we conclude then that  $\mathbf{A}_1 = 0$ . Similarly we can prove that  $\mathbf{A}_i = 0$  for  $i = 2, 3, 4$ .  $\square$

**Corollary 6.9.** *The complexifications of the subgroups  $G_{(3,3,3)}$ ,  $G_{(2,4,4)}$  and  $G_{(2,3,6)}$  of  $PO(3, 1)$  in  $PU(3, 1)$ , have no deformations in  $PU(3, 1)$ .*

## REFERENCES

- [1] V. I. Arnold. *Mathematical Methods of Classical Mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition edition, 1989.
- [2] W. Ballmann, M. Gromov, and V. Schroeder. *Manifolds of Non-positive Curvature*, volume 61 of *Progress in Mathematics*. Birkhauser, Boston, Basel, Stuttgart, 1985.
- [3] R. Benedetti and C. Petronio. *Lectures on Hyperbolic Geometry*. Springer Verlag, Berlin, 1990.
- [4] Élie Cartan. Sur le Groupe de la Géométrie Hypersphérique. *Commentari Mathematici Helvetici*, 4:158–171, 1932.
- [5] Alberto Castro. *Tetrahedral Groups in Complex Hyperbolic Space*. PhD thesis, University of Utah, 1995.
- [6] Alberto Castro. Representations of triangle groups in the Heisenberg group. (submitted for publication), 1997.
- [7] H. S. M. Coxeter. *Regular Complex Polytopes*. Cambridge University Press, Cambridge, second edition, 1991.
- [8] H. S. M. Coxeter and W. O. J. Moser. *Generators and Relations for Discrete Groups*. Springer-Verlag, Berlin, 1965.
- [9] William M. Goldman. Complex Hyperbolic Geometry. preprint, 1994.
- [10] William M. Goldman and John R. Parker. Complex Hyperbolic Ideal Triangle Groups. *Journal für die reine und angewandte Mathematik*, 425:71–86, 1992.
- [11] William M. Goldman and John R. Parker. Dirichlet Polyhedra for Dihedral Groups acting on Complex Hyperbolic Space. *The Journal of Geometric Analysis*, 2(6):517–554, 1992.
- [12] H. Jacobowitz. *An Introduction to CR Structures*, volume 32 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1990.
- [13] G. Mostow. *Strong rigidity of locally symmetric spaces*, volume 78 of *Annals of Mathematics Studies*. Princeton University Press, New Jersey, 1973.
- [14] G. D. Mostow. On a Remarkable Class of Polyhedra in Complex Hyperbolic Space. *Pacific Journal of Mathematics*, 86:171–276, 1980.
- [15] John G. Ratcliffe. *Foundations of Hiperbolic Manifolds*. Springer, New York, 1994.
- [16] William P. Thurston. The Geometry and Topology of Three Manifolds. preprint, 1979.
- [17] Domingo Toledo. Representations of Surface Groups in Complex Hyperbolic Space. *Journal of Differential Geometry*, 29:125–133, 1989.