Math 13, Fall 2000
Last Test

Instructions
Work all problems, and write up your answers neatly with complete justification for your assertions. Individual work is expected on this exam. You may consult with Alberto, but with no one else. You may use your text and notes, and calculators – in short, any inanimate sources. Cite any sources you do use. If there are any typos or other changes on the exam, they will be posted on the web site and an e-mail will be sent to the class (so check your e-mail). The exam is due at the beginning of class of Friday, December 8.

Problems

1. If you attempt to use the formula
   \[ A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1} \, dA \]
   to find the area of the top half of the sphere \( x^2 + y^2 + z^2 = r^2 \) you have a problem because the double integral is improper. In fact, the integrand has an infinite discontinuity at every point of the boundary circle \( x^2 + y^2 = r^2 \). However, the integral can be computed as the limit of the integral over the disk \( x^2 + y^2 \leq t^2 \) as \( t \to r^- \). Use this method to show that the area of a sphere of radius \( r \) is \( 4\pi r^2 \).

2. Evaluate the iterated integral:
   \[ \int_0^8 \int_{\sqrt{y}}^2 e^{x^4} \, dx \, dy \]

3. Find the volume inside both the cylinder \( x^2 + y^2 = 4 \) and the ellipsoid \( 4x^2 + 4y^2 + z^2 = 64 \).

4. A lamina occupies the region inside the circle \( x^2 + y^2 = 2y \) but outside the circle \( x^2 + y^2 = 1 \). Find the center of mass if the density at any point is inversely proportional to its distance from the origin.

5. Describe a region whose area is given by the integral
   \[ \int_0^\pi \int_0^{1+\sin \theta} r \, dr \, d\theta \]

6. Evaluate the integral
   \[ \iint_R (x + y) \, dA, \]
   where \( R \) is the square with vertices \((0,0), (2,3), (5,1), \) and \((3,-2)\). (Hint: Make an appropriate change of variables.)
7. Evaluate the line integrals

(a) \[ \int_C x^3 y \, ds \]  

Where \( C \) is the sine curve: \( y = \sin x \), \( 0 \leq x \leq \pi/2 \).

(b) \[ \int_C y \, dx + z \, dy + x \, dz \]  

Where \( C \) consists of the line segments from \((0, 0, 0)\) to \((1, 1, 2)\) and from \((1, 1, 2)\) to \((3, 1, 4)\).