Topics in Curve and Surface Implicitization

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Outline

Curves:

- Moving Lines & μ-Bases
- Moving Curve Ideal & the Rees Algebra
- Adjoint Curves

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- Moving Lines & μ-Bases
- Moving Curve Ideal & the Rees Algebra
- Adjoint Curves

Surfaces:

- Parametrized Surfaces
- Moving Planes & Syzygies
- Affine, Projective & Bihomogeneous
- The Resultant of a μ-Basis

Curve Implicitization

Turn a *parametrization* into an *equation*.

- ***Affine***: Turn

$$x = \frac{a(t)}{c(t)}, \quad y = \frac{b(t)}{c(t)}$$

into $F(x, y) = 0$.

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into $F(x,y) = 0$.

- ***Projective***: Turn

$$x = a(s,t), \;\; y = b(s,t), \;\; z = c(s,t)$$

into $F(x,y,z) = 0$.

Moving Lines

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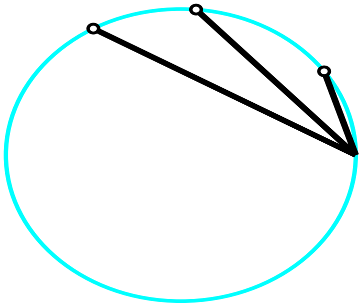
- If two moving lines follow a parametrization, their intersection *is* the parametrization.

Moving Line Picture

Here are two moving lines for an ellipse:



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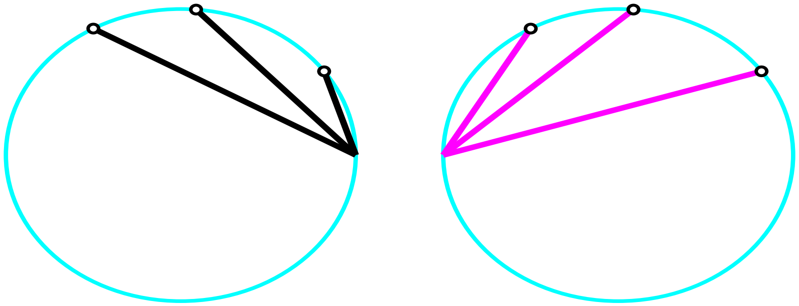

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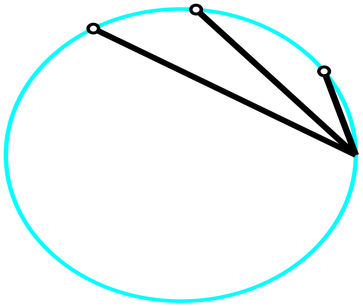
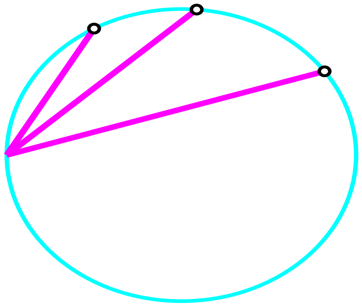
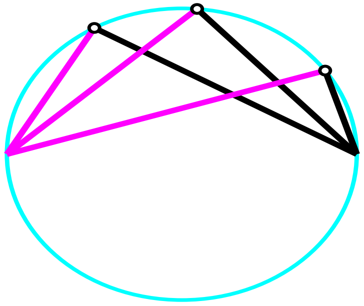




Together they *define* the ellipse:



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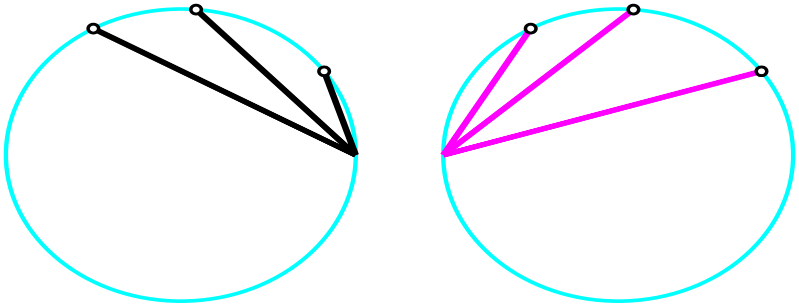

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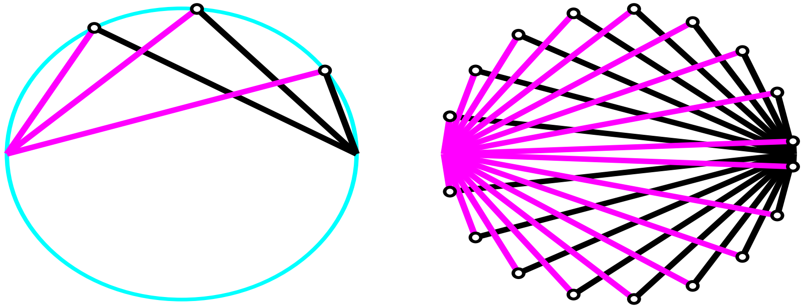

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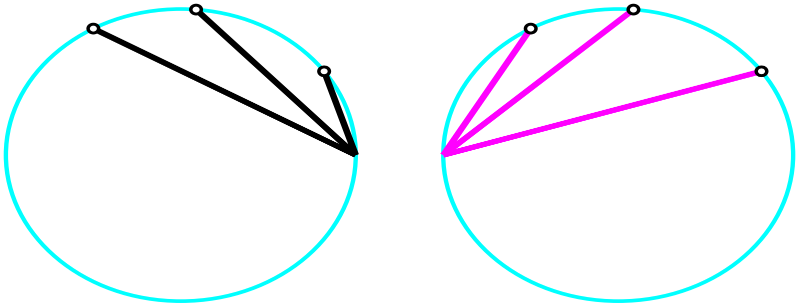

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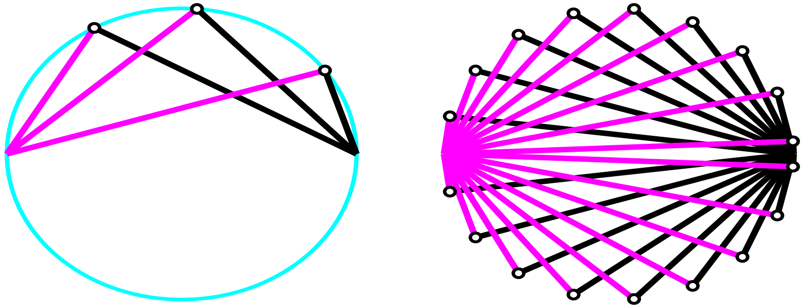

This construction is due Steiner in 1832.

μ-Bases

Theorem: $a, b, c \in k[s,t]$ homogeneous, deg $= n$, $\gcd(a,b,c) = 1$. There exist moving lines p, q with:

1. Every moving line can be uniquely written
$$up + vq,$$
where u and v are homogeneous of degree $m - \deg(p)$ and $m - \deg(q)$.
2. $\deg(p) + \deg(q) = n$.

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2. $\deg(p) + \deg(q) = n$.

Definition: p, q are a *μ-basis* when

$$\mu = \deg(p) \leq \deg(q) = n - \mu.$$

Proof

Let $R = k[s,t]$ and $I = \langle a,b,c \rangle \subset R$.

The *Hilbert Syzygy Theorem* and a *Hilbert polynomial* computation give a free resolution

$$0 \to R(-n-\mu) \oplus R(-2n+\mu) \to R(-n)^3 \xrightarrow{(a,b,c)} I \to 0.$$

The kernel of (a,b,c) is $\mathrm{Syz}(a,b,c)$. It consists of all triples $(A,B,C) \in R^3$ such that

$$Aa + Bb + Cc = 0.$$

These give the moving lines $Ax + By + Cz$

that follow the curve. QED

Some History

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- 1890 – Hilbert proves the Syzygy Theorem and proves Meyer's conjecture.

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- 1890 – Hilbert proves the Syzygy Theorem and proves Meyer's conjecture.
- 1995 – Sederberg and Chen interpret moving lines in terms of syzygies.

Moving Curves

Moving lines are not the full story. Let $R = k[s,t]$.

- A polynomial

$$F = \sum_{i+j+l=m} A_{ijl}(s,t)\, x^i y^j z^l \in R[x,y,z]$$

is called a *moving curve* of degree m.

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- The *moving curve ideal* $MC \subset R[x,y,z]$ is generated by these moving curves.

The Rees Algebra

$I = \langle a, b, c \rangle \subset R = k[s, t]$ has *Rees algebra*

$$R[I] = \bigoplus_{i=0}^{\infty} I^i e^i \subset R[e].$$

Rees algebras are important in commutative algebra.

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$$R[I] = \bigoplus_{i=0}^{\infty} I^i e^i \subset R[e].$$

Rees algebras are important in commutative algebra.

The map $(x, y, z) \mapsto (ae, be, ce)$ gives a surjection

$$R[x, y, z] \longrightarrow R[I].$$

The kernel is MC. Thus *the moving curve ideal gives the defining relations of the Rees algebra!*

Example 1

$$a = 6s^2t^2 - 4t^4, \quad b = 4s^3t - 4st^3, \quad c = s^4$$

The moving curve ideal has five generators:

- Two moving lines of degree 2 in s, t:

  $p = st\,x + (\frac{1}{2}s^2 - t^2)y - 2st\,z$

  $q = s^2\,x - st\,y - 2t^2\,z$

- Two moving conics of degree 1 in s, t:

  $s\,xy - t\,y^2 - 2t\,xz - s\,yz + 4t\,z^2$

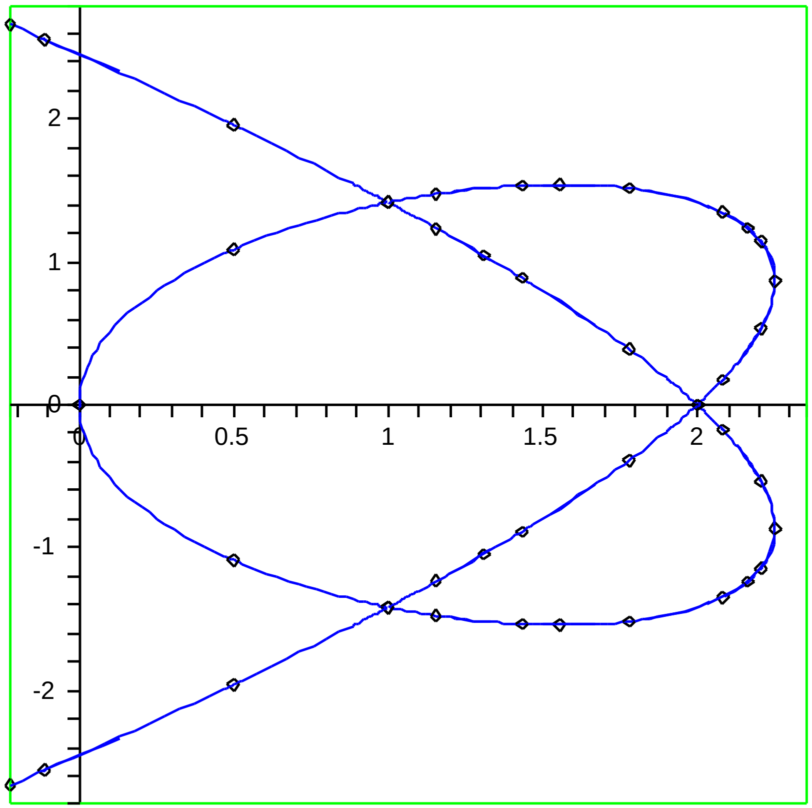
  $s\,x^2 - t\,xy + \frac{1}{2}s\,y^2 - 2s\,xz + t\,yz$

- The implicit equation:

  $y^4 + 4x^3z + 2xy^2z - 16x^2z^2 - 6y^2z^2 + 16xz^3$

Example 1 Picture

Here is the curve of Example 1:



Compute Example 1

To generate MC, begin with the moving lines:

$$p = st\,x + (\tfrac{1}{2}s^2 - t^2)y - 2st\,z$$
$$q = s^2\,x - st\,y - 2t^2\,z$$

- s, t^2 give

$$p = (tx + \tfrac{1}{2}sy - 2tz)s + (-y)t^2$$
$$q = (sx - ty)s + (-2z)t^2$$

The *Sylvester form* is

$$\det\begin{pmatrix} tx + \frac{1}{2}sy - 2tz & -y \\ sx - ty & -2z \end{pmatrix}$$

This is the first moving conic generator of MC!

Compute Example 1

- s^2, t give the second moving conic generator!
- s, t and the moving conic generators give

$$(xy - yz)s + (4z^2 - y^2 - 2xz)t$$
$$(x^2 + \tfrac{1}{2}y^2 - 2xz)s + (yz - xy)t$$

The Sylvester form is

$$\det\begin{pmatrix} xy - yz & 4z^2 - y^2 - 2xz \\ x^2 + \frac{1}{2}y^2 - 2xz & yz - xy \end{pmatrix}$$

This is the implicit equation!

- The implicit equation is also $\mathrm{Res}(p, q)$.

Example 2

$$a = 3s^3t - 3s^2t^2, \quad b = 3s^2t^2 - 3st^3, \quad c = (s^2 + t^2)^2$$

The moving curve ideal has five generators:

- Two moving lines of degree 1,3 in s, t:

  $p = t\,x - s\,y$

  $q = s^3\,x + (2s^2t + t^3)y + (3st^2 - 3s^2t)z$

- One moving conic of degree 2 in s, t:

  $s^2\,x^2 + (2s^2 + t^2)y^2 + (3st - 3s^2)yz$

- One moving cubic of degree 1 in s, t:

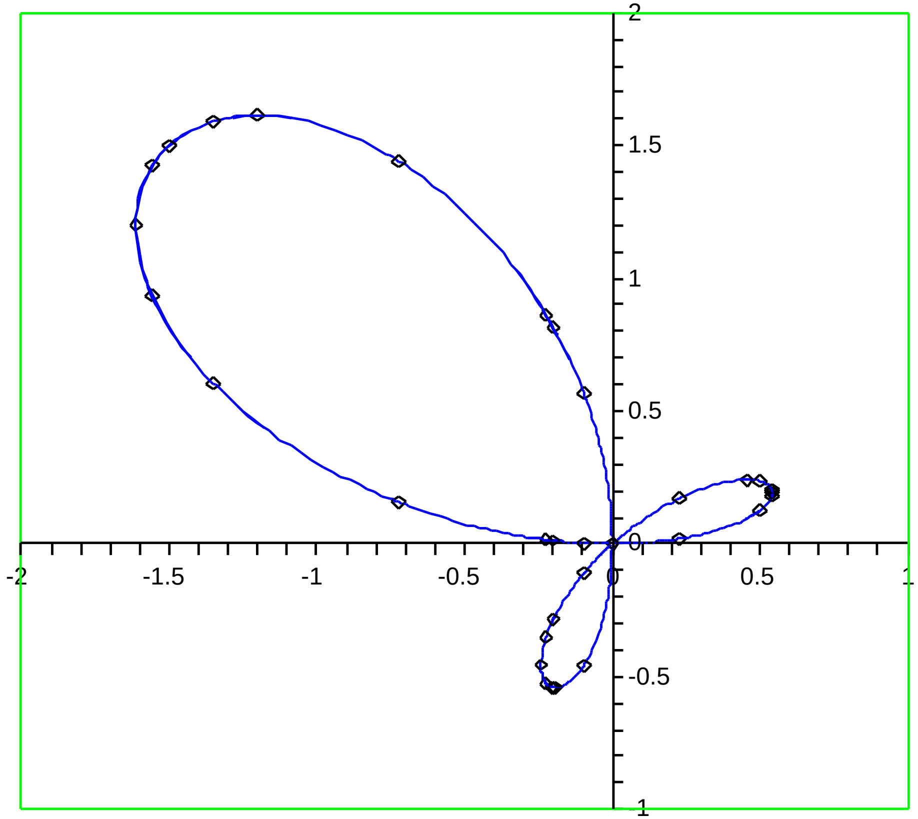
  $s\,x^3 + 2s\,xy^2 + t\,y^3 - 3s\,xyz + 3s\,y^2z$

- The implicit equation:

  $x^4 + 2x^2y^2 + y^4 - 3x^2yz + 3xy^2z$

Example 2 Picture

Here is the curve of Example 2:



Compute Example 2

To generate MC, begin with the moving lines:

$$p = t\,x - s\,y$$
$$q = s^3\,x + (2s^2t + t^3)y + (3st^2 - 3s^2t)z$$

- The moving lines give

$$p = (-y)s + (x)t$$
$$q = (s^2x)s + ((2s^2 + t^2)y + (3st - 3s^2)z)t$$

The Sylvester form is

$$\det\begin{pmatrix} -y & x \\ s^2x & (2s^2+t^2)y + (3st-3s^2)z \end{pmatrix}$$

This is -1 times the moving conic generator!

Compute Example 2

- $p = t\,x - s\,y$ and the moving conic give
$$\det\begin{pmatrix} -y & x \\ sx^2 + 2sy^2 - 3syz & ty^2 + 3syz \end{pmatrix}$$
This is -1 times the moving cubic generator!
- $p = t\,x - s\,y$ and the moving cubic give
$$\det\begin{pmatrix} -y & x \\ x^3 + 2xy^2 - 3xyz + 3y^2z & y^3 \end{pmatrix}$$
This is -1 times is the implicit equation!
- The implicit equation is also $\mathrm{Res}(p,q)$.

Theorems

There are theorems that explain these examples, plus results on parametrized curves in $\mathbb{P}^n$.

Some of the people involved:

C-
C-, Hoffman, and Wang
Busé
Hoon, Simis, and Vasconcelos
Kustin, Polini, and Ulrich
Cortadellas Benítez and D'Andrea
Goldman, Jia, and Wang

I will explain some of this tomorrow.

Rational Plane Curves

Theorem: If $C \subset \mathbb{P}^2$ is defined by an irreducible equation of degree n, then C is rational $\iff$

$$(n-1)(n-2) = \sum_p \nu_p(\nu_p - 1),$$

where the sum is over all singular points p of C and ν_p is the multiplicity of C at p.

Classical Proof: We will use *adjoint curves*. A curve D of degree m is *adjoint* to C if:

- At all singular points p of C with multiplicity ν_p, the curve D has multiplicity at least $\nu_p - 1$.

The Classical Proof

Lemma: For $m \in \{n-1, n-2\}$, $\exists$ a 1-dim linear system of plane curves whose general member D is adjoint to C and meets C in $mn - (n-1)(n-2) - 1$ fixed smooth points of C.

Consequences:

- By Bezout, $mn = D \cdot C = \sum_p (\nu_p - 1)\nu_p + mn - (n-1)(n-2) - 1 \ + $ one more point.

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Also: we get the parametrization by a resultant.

Example

Consider the affine curve defined by

$$F(x,y) = y^4 + 4x^3 + 2xy^2 - 16x^2 - 6y^2 + 16x = 0.$$

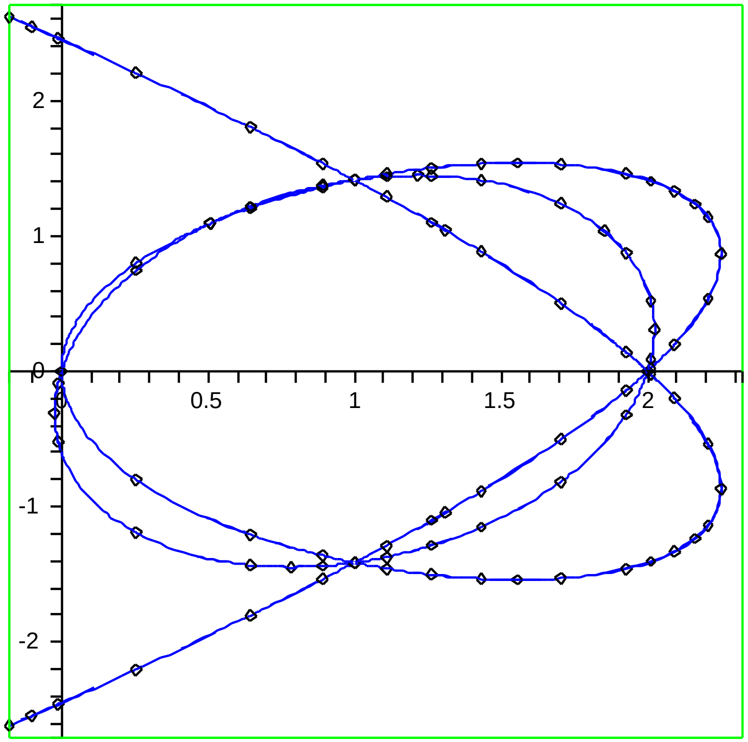
Note $(4-1)(4-2) = 3$ [sing pts] $\cdot\, 2$ [mult] $\cdot\, (2-1)$.
To parametrize, use the linear system of conics

$$G_{s,t}(x,y) = sx^2 - txy + \tfrac{1}{2}sy^2 - 2sx + ty = 0.$$

These adjoint curves all go through the origin.

Observation: G is one of our moving conics!

Picture



These conics go through the singular points and the origin, plus one more point that moves.

The Parametrization

Compute the resultants:

$$\mathrm{Res}(F, G, y) = x(x-1)^4(x-2)^2(s^4x - 6t^2s^2 + 4t^4)$$

$$\mathrm{Res}(F, G, x) = y^3(y^2-2)^2(s^3y - 4ts^2 + 4t^3)$$

The constant factors show that $G = 0$ goes through the origin and the singular points of $F = 0$. The other factors give

$$x = \frac{6s^2t^2 - 4t^4}{s^4}, \; y = \frac{4s^3t - 4st^3}{s^4}.$$

This is an affine version of our original parametrization of $F = 0$!

Theorem

Theorem: Given a proper parametrization of degree n, there are elements of the moving curve ideal MC of degree one in s, t and degree $n-1$ or $n-2$ in x, y, z can be chosen to be *adjoint linear systems* on the rational curve defined by the parametrization.

- Proof by Busé; $\mu = 1$ by C-, Hoffman, Wang.
- Theorem based on an observation of Sendra.
- Moving lines: small deg x, y, z, large deg s, t. Adjoint curves: large deg x, y, z, small deg s, t.

Surfaces

Example: The Steiner surface is given by

$$
\begin{aligned}
x &= \frac{a(s,t)}{d(s,t)} = \frac{2st}{s^2+t^2+1} \\
y &= \frac{b(s,t)}{d(s,t)} = \frac{2t}{s^2+t^2+1} \\
z &= \frac{c(s,t)}{d(s,t)} = \frac{2s}{s^2+t^2+1}
\end{aligned}
$$

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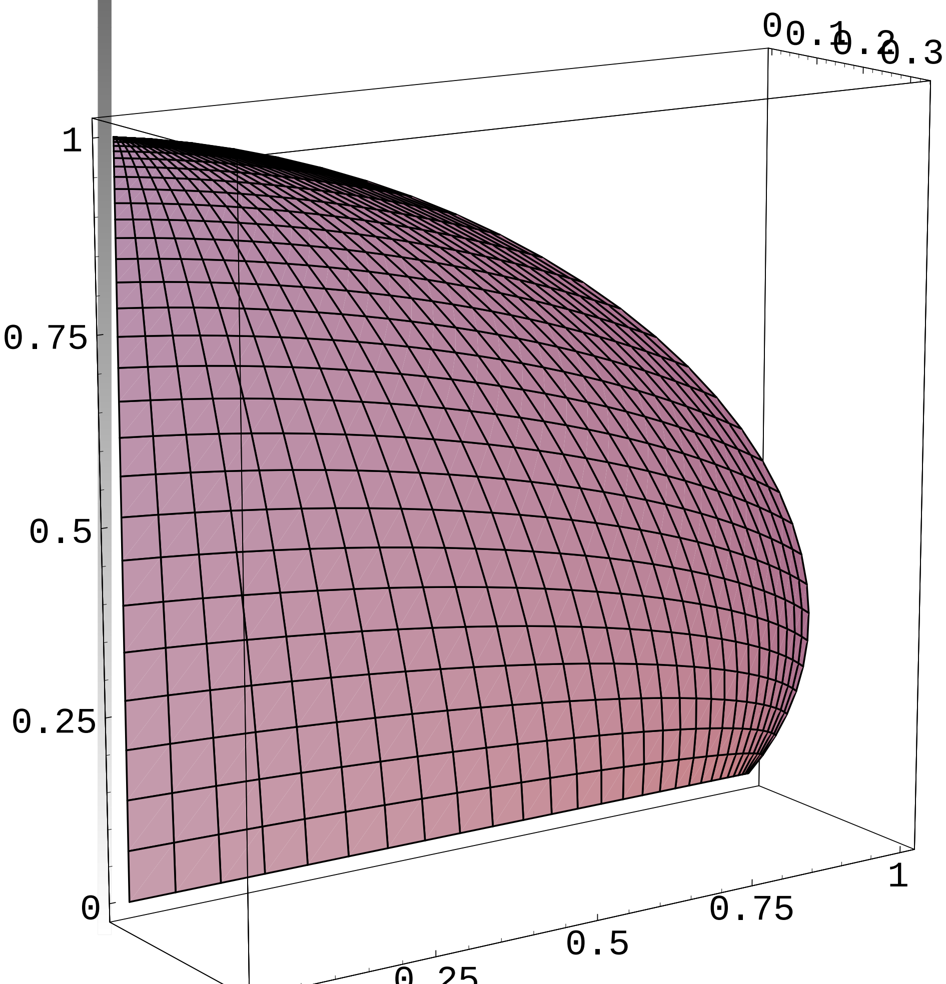

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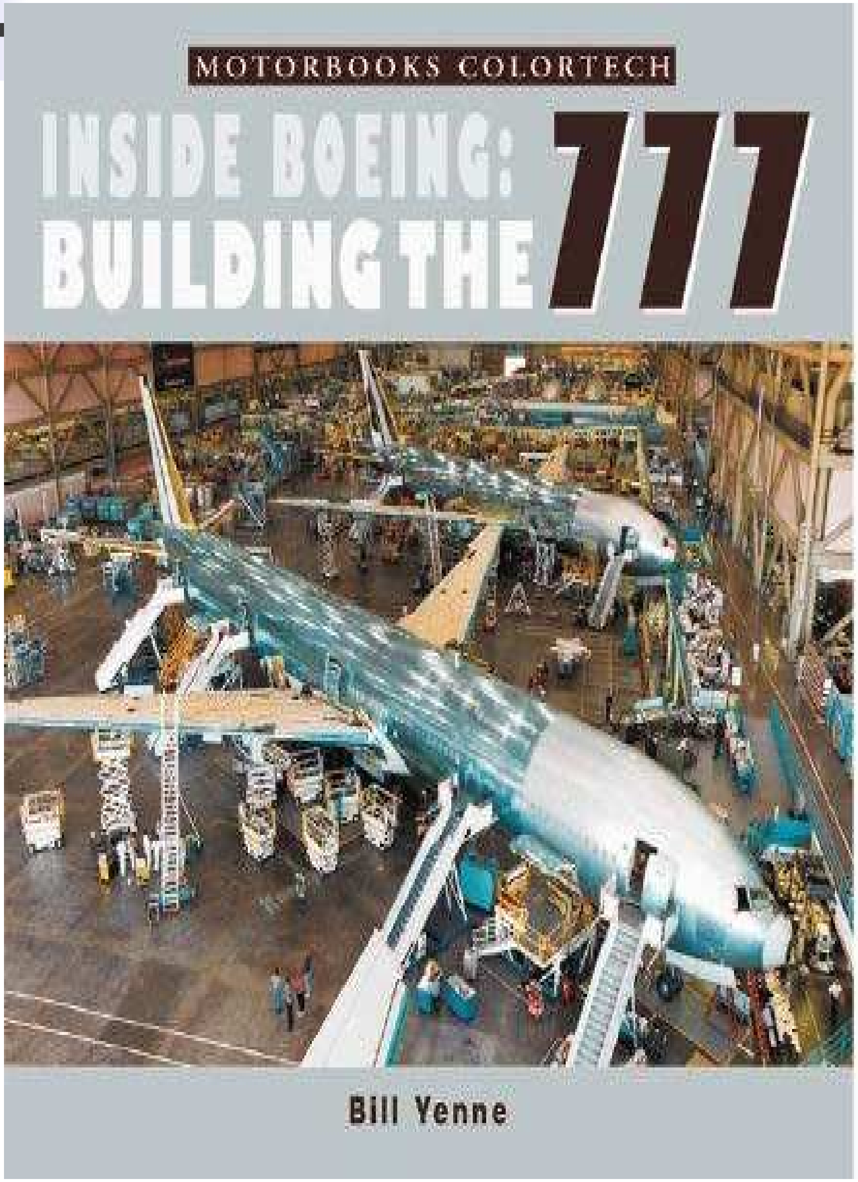
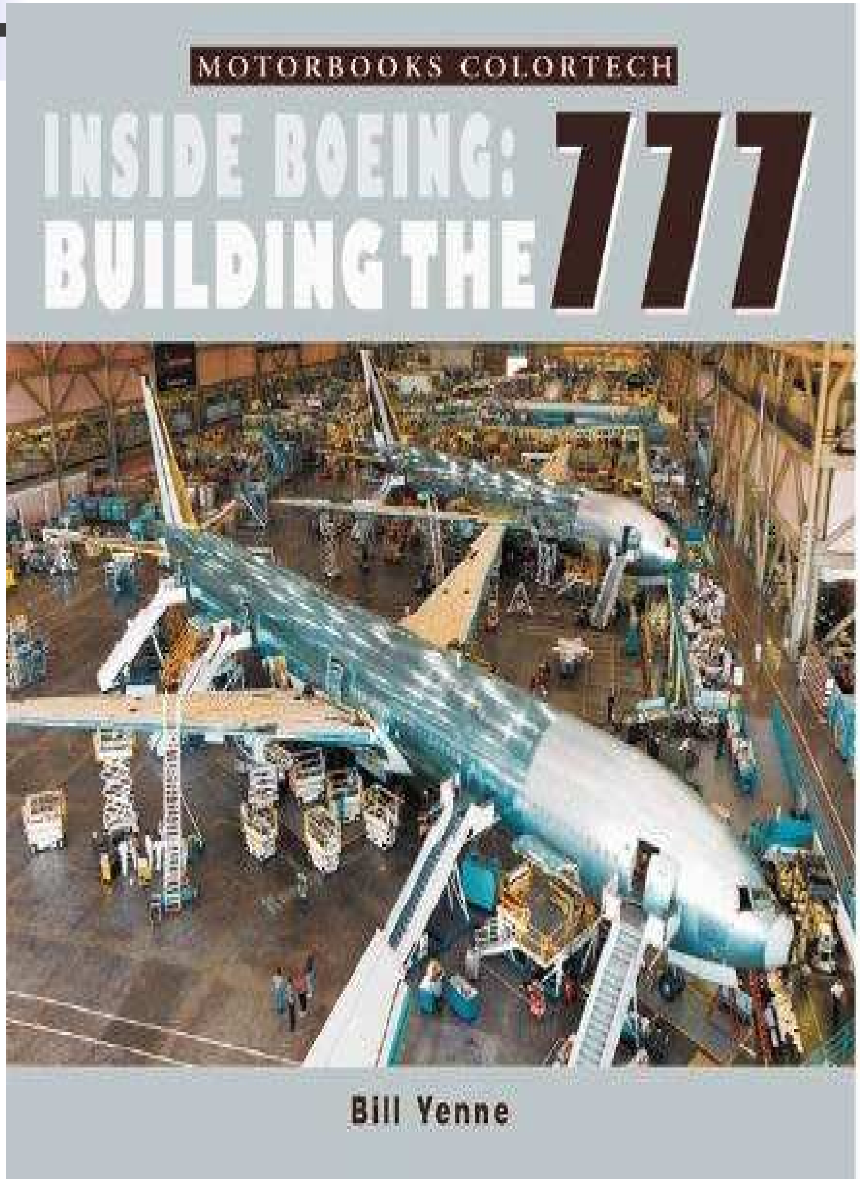
Boeing 777


MOTORBOOKS COLORTECH
INSIDE BOEING: BUILDING THE 777
Bill Yenne

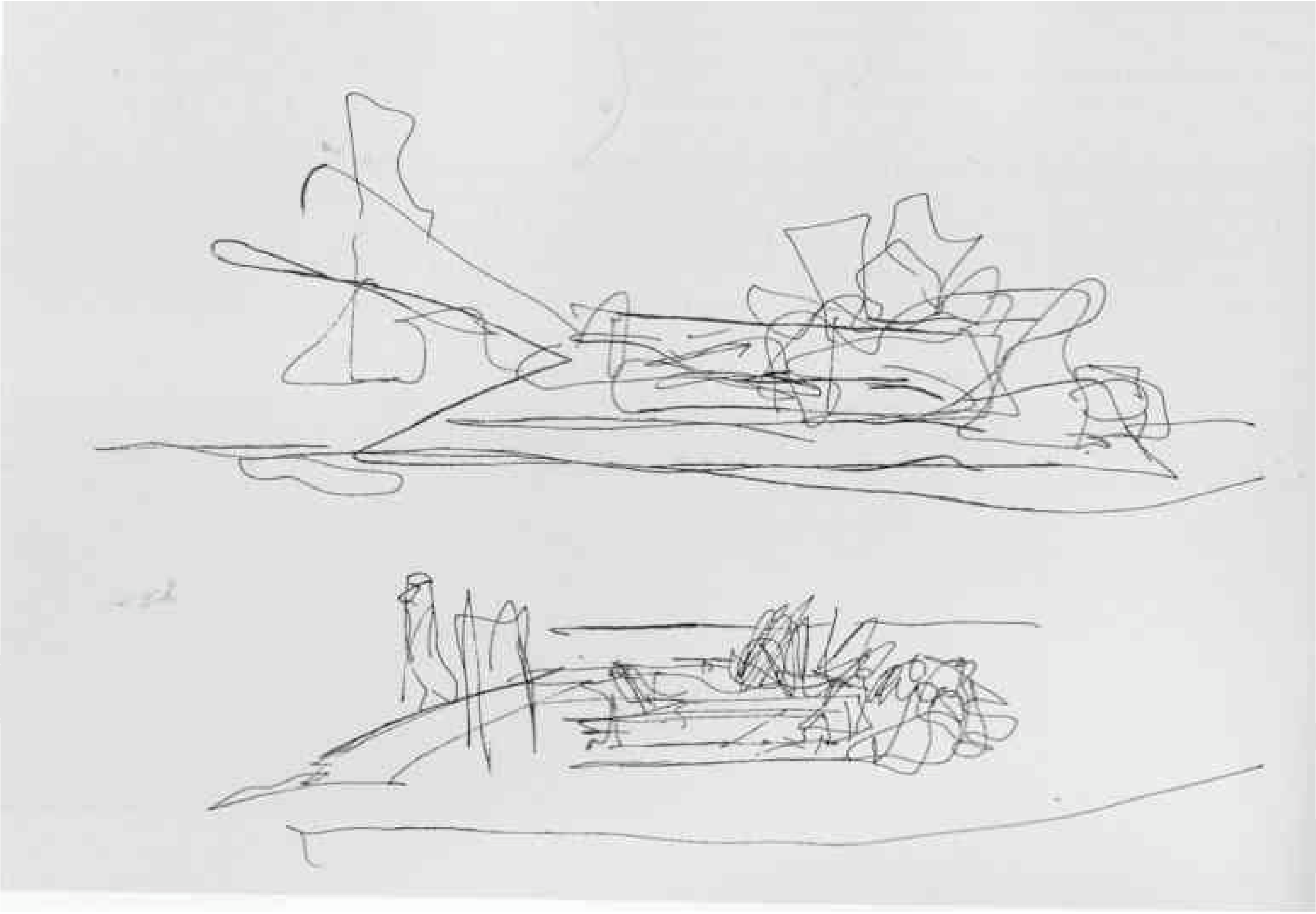
Boeing 777



The Boeing 777 was designed using 50 million surface patches.

Guggenheim Bilbao

Gehry Sketch



Bilbao Close-Up

Commutative Algebra

Affine Case: A moving plane

$$Ax + By + Cz + D = 0, \quad A, B, C, D \in k[s,t]$$

follows a, b, c, d iff $Aa + Bb + Cc + Dd = 0$. Thus moving planes live in the syzygy module

$$\mathrm{Syz}(a, b, c, d) \subset k[s,t]^4.$$

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Theorem: The syzygy module $\mathrm{Syz}(a, b, c, d)$ is a free $k[s,t]$-module of rank 3.

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Proof: Auslander-Buchsbaum & Quillen-Suslin!

Commutative Algebra

Projective Case: More complicated!

- $I = \langle a, b, c, d \rangle \subset R = k[s, t, u]$ homogeneous
- $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ rational map
- *Basepoints* $\mathbf{V}(a, b, c, d) \subset \mathbb{P}^2$
- $S = \overline{\text{image}} \subset \mathbb{P}^3$ parametrized surface
- $\deg \phi \cdot \deg S = n^2 - \sum_p e_p$
- $e_p =$ *Hilbert-Samuel Multiplicity*

Projective Case

The following are equivalent:

- $\mathrm{Syz}(a,b,c,d)$ is free
- $\mathrm{pd}(R/I) = 2$
- R/I is Cohen-Macaulay
- I is saturated.

Example: Cubic surface in $\mathbb{P}^3$ has a,b,c,d deg 3:

- $\mathrm{Syz}(a,b,c,d)$: 3 moving planes deg 1 in s,t,u.
- Basepoints: Six.

Also: No basepoints $\Rightarrow \mathrm{Syz}(a,b,c,d)$ not free.

The Bihomogeneous Case

Geometric Modeling often uses *rectangular* surfaces patches, built from polynomials in s, t whose Newton polygon is a rectangle.

This leads naturally to a parametrization

$$\phi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

(assuming no basepoints), where ϕ is given by *bihomogeneous* polynomials of bidegree (n, m).

Bigraded commutative algebra is *very different!* I will give an example on Thursday.

The Affine Case

A basis of $\mathrm{Syz}(a, b, c, d)$ over $k[s, t]$ is a μ-basis.

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Write the μ-basis as

$$\begin{aligned} p = A\ x + B\ y + C\ z + D &= 0 \\ q = A'\,x + B'\,y + C'\,z + D' &= 0 \\ r = A''x + B''y + C''z + D'' &= 0 \end{aligned}$$

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By Cramer, a, b, c, d are the 3×3 minors of

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ A' & B' & C' & D' \\ A'' & B'' & C'' & D'' \end{pmatrix}$$

Resultant of a μ-Basis

For surfaces, the resultant of an affine μ-basis *almost* gives the implicit equation.

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Analysis: At a point (x, y, z) where

$$\mathrm{Res}(p, q, r) = 0,$$

the equations

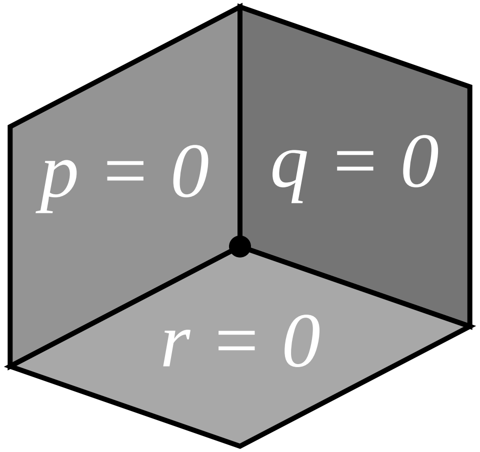
$$\begin{aligned} p = A\ x + B\ y + C\ z + D &= 0 \\ q = A'\, x + B'\, y + C'\, z + D' &= 0 \\ r = A'' x + B'' y + C'' z + D'' &= 0 \end{aligned}$$

have a solution s, t (possibly at ∞).

No Basepoints

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ A' & B' & C' & D' \\ A'' & B'' & C'' & D'' \end{pmatrix}$$

has rank 3 since a, b, c, d are the 3×3 minors. So *no basepoints* $\Rightarrow$ *the moving planes always have a unique point of intersection!*



Basepoints

At a basepoint, the parameter values "blow up" to an *exceptional curve* on the surface. These curves come in three flavors:

- A line.
- A plane curve.
- A space curve.

Basepoints

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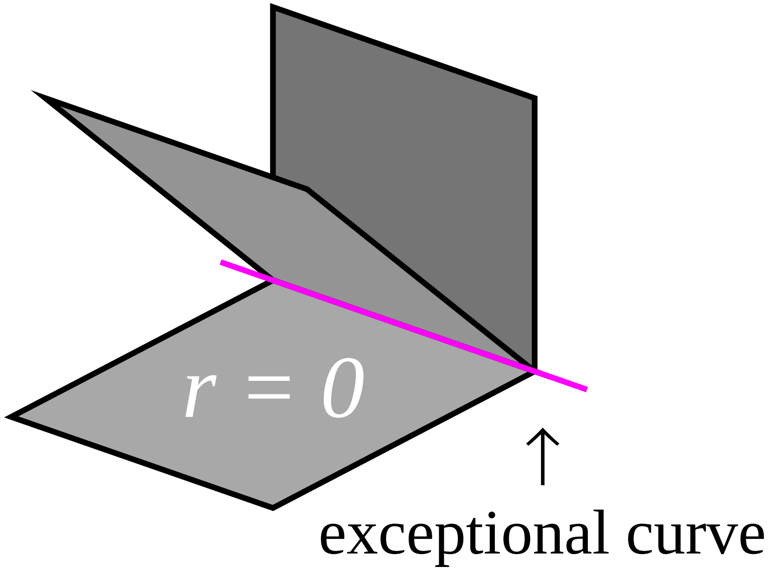
These cases correspond to the *rank* of

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ A' & B' & C' & D' \\ A'' & B'' & C'' & D'' \end{pmatrix}.$$

Rank 2 Basepoints

Here, the moving planes intersect in a line:

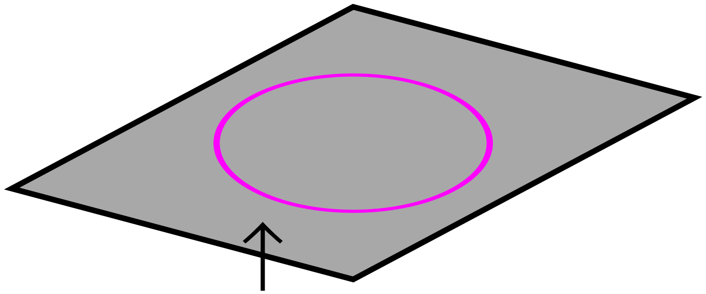


Furthermore:

- The resultant $\mathrm{Res}(p, q, r)$ vanishes *exactly* on the surface, at least for s, t finite.
- A basepoint has rank two $\iff$ it is LCI!

Rank 1 Basepoints

Here, the moving planes coincide:



Furthermore:

- The resultant $\mathrm{Res}(p, q, r)$ has an extraneous factor $=$ the equation of the plane to the power $e_p - d_p$, $d_p = \dim_k \mathcal{O}_p/\langle a, b, c, d\rangle$.
- A basepoint has rank one $\iff$ $\langle a, b, c, d\rangle$ is almost LCI.

Rank 0 Basepoints

Here, the moving "planes" are the ambient space, since we have a space curve. Thus:

- The resultant $\mathrm{Res}(p, q, r)$ vanishes identically.
- A basepoint has rank zero $\iff$ locally $\langle a, b, c, d\rangle$ requires four generators.

Hence

$$\mathrm{Res}(p, q, r)$$

requires a truly bad basepoint before it vanishes!