Introduction to Algebraic Geometry

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Acknowledgements

These notes are intended to help students make the transition between the elementary aspects of algebraic geometry (varieties in affine and projective space, etc.) and some of its more sophisticated aspects (normal varieties, Weil and Cartier divisors, etc.).

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§1. Affine Varieties

Basic Definitions. For simplicity, we will work over the complex numbers $\mathbb{C}$. Then, given polynomials $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_n]$, we get the *affine variety*

$$\mathbf{V}(f_1, \ldots, f_s) = \{a \in \mathbb{C}^n \mid f_1(a) = \cdots = f_s(a) = 0\}.$$

More generally, if $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is an ideal, then we define

$$\mathbf{V}(I) = \{a \in \mathbb{C}^n \mid f(a) = 0 \text{ for all } f \in I\}.$$

Exercise 1.1. Let $I = \langle f_1, \ldots, f_s \rangle \subset \mathbb{C}[x_1, \ldots, x_n]$ be the ideal generated by $f_1, \ldots, f_s$. Show that $\mathbf{V}(I) = \mathbf{V}(f_1, \ldots, f_s)$. (All ideals in $\mathbb{C}[x_1, \ldots, x_n]$ are of this form by the Hilbert Basis Theorem.)

Conversely, given an affine variety $V \subset \mathbb{C}^n$, we get the ideal

$$\mathbf{I}(V) = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(a) = 0 \text{ for all } a \in V\}.$$

Exercise 1.2. Let $V \subset \mathbb{C}^n$ be an affine variety and $I \subset \mathbb{C}[x_1, \ldots, x_n]$ an ideal. Show that:

a. $V = \mathbf{V}(\mathbf{I}(V))$.
b. $I \subset \sqrt{I} \subset \mathbf{I}(\mathbf{V}(I))$, where $\sqrt{I} = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f^m \in I,\ m \geq 1\}$ is the *radical* of I.

The above exercise actually works over any field. But since $\mathbb{C}$ is algebraically closed, we also have the following basic result of Hilbert.

Hilbert Nullstellensatz. *For any ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$, we have $\sqrt{I} = \mathbf{I}(\mathbf{V}(I))$.*

A proof can be found in Chapter 4 of [3]. This theorem allows us to translate algebra into geometry and vice versa. Here is an example.

Exercise 1.3. Use the Nullstellensatz to show the following.

a. Every maximal ideal of $\mathbb{C}[x_1, \ldots, x_n]$ is of the form $\langle x_1 - a_1, \ldots, x_n - a_n \rangle$, where $a_i \in \mathbb{C}$. Thus there is a one-to-one correspondence between points of $\mathbb{C}^n$ and maximal ideals of $\mathbb{C}[x_1, \ldots, x_n]$.
b. An ideal I is *radical* if $I = \sqrt{I}$. Show that the correspondence of part a extends to a one-to-one correspondence

$$\text{affine varieties of } \mathbb{C}^n \longleftrightarrow \text{radical ideals of } \mathbb{C}[x_1, \ldots, x_n].$$

Coordinate Rings. We next consider polynomial functions on an affine variety V. Note that two polynomials $f, g \in \mathbb{C}[x_1, \dots, x_n]$ give the same function on V if and only if their difference lies in $\mathbf{I}(V)$. Thus the ring of such functions is naturally isomorphic to the quotient ring

$$\mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_n]/\mathbf{I}(V).$$

This ring is called the *coordinate ring* of V. There is a close relation between V and $\mathbb{C}[V]$. The following two exercises explore aspects of this relation.

Exercise 1.4. Two affine varieties $V_1 \subset \mathbb{C}^n$ and $V_2 \subset \mathbb{C}^m$ are *isomorphic* if there are polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^m$ and $G : \mathbb{C}^m \to \mathbb{C}^n$ such that $F(V_1) = V_2$, $G(V_2) = V_1$, and the compositions $F \circ G$ and $G \circ F$ are the identity when restricted to V_2 and V_1 respectively. Prove that two affine varieties are isomorphic if and only if their coordinate rings are isomorphic $\mathbb{C}$-algebras.

Exercise 1.5. Let $V \subset \mathbb{C}^n$ be an affine variety.

a. Given $a = (a_1, \dots, a_n) \in \mathbb{C}^n$, show that $a \in V$ if and only if $\mathbf{I}(V) \subset \langle x_1 - a_1, \dots, x_n - a_n \rangle$.
b. Conclude that there is a one-to-one correspondence between points of affine variety V and maximal ideals of its coordinate ring $\mathbb{C}[V]$.

We can characterize coordinate rings of affine varieties as follows.

Proposition 1.1. *A $\mathbb{C}$-algebra R is isomorphic to the coordinate ring of an affine variety if and only if R is a finitely generated $\mathbb{C}$-algebra with no nonzero nilpotents (i.e., if $f \in R$ satisfies $f^m = 0$, then $f = 0$).*

Proof. If $R = \mathbb{C}[V]$ for $V \subset \mathbb{C}^n$, then we need only show that R has no nonzero nilpotents. This is easy, for if $f \in \mathbb{C}[x_1, \dots, x_n]$ and f^m vanishes on V, then so does f. Thus $\mathbf{I}(V)$ is radical, which means that $\mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_n]/\mathbf{I}(V)$ has no nonzero nilpotents.

Conversely, R finitely generated as a $\mathbb{C}$-algebra implies that there is a surjective homomorphism $\varphi : \mathbb{C}[x_1, \dots, x_n] \to R$. Let $I = \ker \varphi$, and note that $I = \sqrt{I}$ since R has no nonzero nilpotents. Then let $V = \mathbf{V}(I) \subset \mathbb{C}^n$. The coordinate ring of V is $\mathbb{C}[x_1, \dots, x_n]/\mathbf{I}(V)$. Using the Nullstellensatz, we see that $\mathbf{I}(V) = \mathbf{I}(\mathbf{V}(I)) = \sqrt{I} = I$. Thus $\mathbb{C}[V]$ is isomorphic to R. $\square$

To emphasize the close relation between V and $\mathbb{C}[V]$, we will sometimes write

$$V = \mathrm{Spec}(\mathbb{C}[V]). \tag{1.1}$$

Furthermore, this can be made canonical by identifying V with the set of maximal ideals of $\mathbb{C}[V]$ via Exercise 1.5. This is part of a general contruction in algebraic geometry which takes any commutative ring R and defines the *affine scheme* Spec(R). The general definition of Spec uses all prime ideals of R and not just the maximal ideals as we have done.* Readers wishing to learn more about schemes should consult [4] and [5].

Subvarieties and the Zariski Topology. Given an affine variety $V \subset \mathbb{C}^n$, a subset $W \subset V$ is a *subvariety* if W is also an affine variety. This easily implies that $\mathbf{I}(V) \subset \mathbf{I}(W)$. In terms of the coordinate ring $R = \mathbb{C}[V]$, we conclude that there is a one-to-one correspondence

$$\text{subvarieties of } \mathrm{Spec}(R) \longleftrightarrow \text{radical ideals of } R.$$

* Thus (1.1) should be written $V = \mathrm{Specm}(\mathbb{C}[V])$, the maximal spectrum of $\mathbb{C}[V]$.

An affine variety has two interesting topologies. First, we have the induced topology from the usual topology on $\mathbb{C}^n$. This is sometimes called the *classical topology*. The other topology is defined as follows. Given a subvariety $W \subset V$, the complement $V - W$ is called a *Zariski open subset* of V. One easily sees that the Zariski open subsets of V form a topology on V, which is called the *Zariski topology*. Since every subvariety of V is closed in the classical topology (polynomials are continuous), it follows that every Zariski open subset is also open in the classical topology.

Exercise 1.6. Zariski open subsets tend to be large. Here are some examples.

a. Show that the Zariski topology on $\mathbb{C}$ is the *cofinite* topology. This is the topology whose open sets are $\emptyset$ and complements of finite sets.
b. Show that the Zariski topology on $\mathbb{C}^n$ is T_1 but not T_2.

Given a subset $S \subset V$, its closure $\overline{S}$ in the Zariski topology is the smallest subvariety of V containing S. We call $\overline{S}$ the *Zariski closure* of S. It is easy to give examples where this differs from the closure in the classical topology.

Finally, we remark that some Zariski open subsets of an affine variety V are themselves affine varieties. Given $f \in \mathbb{C}[V] - \{0\}$, let $V_f = \{a \in V \mid f(a) \neq 0\} \subset V$.

Lemma 1.2. *V_f is Zariski open in V and has a natural structure as an affine variety.*

Proof. Suppose $V \subset \mathbb{C}^n$ and $\mathbf{I}(V) = \langle f_1, \ldots, f_s \rangle$. Also pick $g \in \mathbb{C}[x_1, \ldots, x_n]$ so that $f = g + \mathbf{I}(V)$. Then $V_f = V - \mathbf{V}(f_1, \ldots, f_s, g)$, so that V_f is Zariski open in V.

Consider a new variable y and let $W = \mathbf{V}(f_1, \ldots, f_s, 1 - gy) \subset \mathbb{C}^n \times \mathbb{C}$. Then $(a, b) \in \mathbb{C}^n \times \mathbb{C}$ lies in W if and only if $a \in V_f$ (and then $b = 1/g(a)$). In other words, the projection map $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ maps W bijectively to V_f. Thus we can identify V_f with the affine variety $W \subset \mathbb{C}^n \times \mathbb{C}$. $\square$

Irreducible Varieties and Rational Functions. An affine variety V is *irreducible* if it cannot be written as union of subvarieties $V = V_1 \cup V_2$ where $V_i \neq V$. We can think of irreducbility in algebraic terms as follows.

Exercise 1.7. Let $V \subset \mathbb{C}^n$ be an affine variety. Prove that V is irreducible $\Leftrightarrow$ $\mathbf{I}(V) \subset \mathbb{C}[x_1, \ldots, x_n]$ is a prime ideal $\Leftrightarrow$ the coordinate ring $\mathbb{C}[V]$ is an integral domain.

Here is an example we will refer to later.

Exercise 1.8. Let $V = \mathbf{V}(xy - zw) \subset \mathbb{C}^4$.

a. Show that $xy - zw$ is irreducible in $\mathbb{C}[x, y, z, w]$.
b. Conclude that $\mathbf{I}(V) = \langle xy - zw \rangle$ and that V is irreducible. Thus the coordinate ring of V is $\mathbb{C}[V] = \mathbb{C}[x, y, z, w]/\langle xy - zw \rangle$.
c. Prove that $\mathbb{C}[V] \simeq \mathbb{C}[ab, cd, ac, bd] \subset \mathbb{C}[a, b, c, d]$. Hint: Prove that V can be parametrized surjectively by $(a, b, c, d) \mapsto (ab, cd, ac, bd)$.

When V is irreducible, the integral domain $\mathbb{C}[V]$ has a field of fractions denoted $\mathbb{C}(V)$. This is the *field of rational functions* on V. For example, when $V = \mathbb{C}^n$, $\mathbb{C}[V]$ is the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{C}(V)$ is the field of rational functions $\mathbb{C}(x_1, \ldots, x_n)$. In general, given $f/g \in \mathbb{C}(V)$, the equation $g = 0$ defines a proper subvariety $W \subset V$ and $f/g : V - W \to \mathbb{C}$ is a well-defined function. This is written $f/g : V \dashrightarrow \mathbb{C}$ and is called a *rational function* on V.

Exercise 1.9. If V is irreducible and $f \in \mathbb{C}[V]$ is nonzero, then the *localization* of $\mathbb{C}[V]$ at f is

$$\mathbb{C}[V]_f = \{g/f^\ell \in \mathbb{C}(V) \mid g \in \mathbb{C}[V],\ \ell \geq 0\}.$$

Prove that $\operatorname{Spec}(\mathbb{C}[V]_f)$ is the affine variety V_f from Lemma 1.2.

An important result is that every affine variety V can be written as a union

$$V = V_1 \cup \cdots \cup V_r$$

where each V_i is irreducible and $V_i \not\subset \bigcup_{j \neq i} V_j$. We call $V_1, \ldots, V_r$ the *irreducible components* of V. The existence and uniqueness of this decomposition is proved in Chapter 4 of [3].

Finally, some references use different terminology. For example, in Hartshorne's book [5], $\mathbf{V}(I) \subset \mathbb{C}^n$ is called an "algebraic set" and the term "affine variety" is reserved for the case when $\mathbf{V}(I)$ is irreducible. We will not use this terminology, though we should point out that our main objects of interest are toric varieties, which are by definition irreducible.

Normal Affine Varieties. Let R be an integral domain with field of fractions K. Then R is *integrally closed* if every element of K which is integral over R (meaning that it is a root of a monic polynomial in $R[x]$) actually lies in R. Here are two examples:

- One can easily show that any UFD is integrally closed.
- The set $\mathcal{O}_K$ of all algebraic integers in a number field K is integrally closed.

Exercise 1.10 below will give an example of an integral domain which is not integrally closed.

Let V be an irreducible affine variety, so that $\mathbb{C}[V]$ is an integral domain. Then V is *normal* if $\mathbb{C}[V]$ is integrally closed. For example, $\mathbb{C}^n$ is normal since its coordinate ring $\mathbb{C}[x_1, \ldots, x_n]$ is a UFD and hence integrally closed. Here is an example of a non-normal affine variety.

Exercise 1.10. Let $C = \mathbf{V}(x^3 - y^2) \subset \mathbb{C}^2$. This is a plane curve with a cusp at the origin.

a. Show that C is irreducible and that $\mathbb{C}[C] = \mathbb{C}[x,y]/\langle x^3 - y^2 \rangle$.

b. Let X and Y be the cosets of x and y in $\mathbb{C}[C]$ respectively. This gives $Y/X \in \mathbb{C}(C)$. Show that $Y/X \notin \mathbb{C}[C]$ and that $(Y/X)^2 = X$.

c. Explain why part b implies that $\mathbb{C}[C]$ is not integrally closed.

Another example is the irreducible variety $V = \mathbf{V}(xy - zw) \subset \mathbb{C}^4$ studied in Exercise 1.8. It is not obvious, but V is normal. This can be proved using the description

$$\mathbb{C}[V] \simeq \mathbb{C}[ab, cd, ac, bd] \subset \mathbb{C}[a, b, c, d]$$

given in part c of Exercise 1.8. The ring $\mathbb{C}[ab, cd, ac, bd]$ is a *semigroup algebra*. Then normality follows from a property called *saturation*.

For us, normality is crucial because toric varieties are all normal. (One can define non-normal toric varieties, but the nicest results hold only in the normal case.)

Finally, any irreducible affine variety V has a *normalization*. To define this, first consider

$$\mathbb{C}[V]' = \{\alpha \in \mathbb{C}(V) : \alpha \text{ is integral over } \mathbb{C}[V]\}.$$

We call $\mathbb{C}[V]'$ the *integral closure* of $\mathbb{C}[V]$. It is easy to see that $\mathbb{C}[V]'$ is integrally closed. With more work, one can also show that $\mathbb{C}[V]'$ is a finitely generated $\mathbb{C}$-algebra (see Theorem 9 on pages 267–268 of [11]). This gives the normal affine variety

$$V' = \operatorname{Spec}(\mathbb{C}[V]')$$

which is the *normalization* of V. Note that the natural inclusion $\mathbb{C}[V] \subset \mathbb{C}[V]' = \mathbb{C}[V']$ corresponds to a map $V' \to V$. This is called the *normalization map*.

Exercise 1.11. Let $C = \mathbf{V}(x^3 - y^2) \subset \mathbb{C}^2$ be the curve considered in Exercise 1.10.

a. Let X and Y have the same meaning as in Exercise 1.10. Show that $\mathbb{C}[Y/X] \subset \mathbb{C}(C)$ is the integral closure of $\mathbb{C}[C]$.
b. Show that the normalization map is the map $\mathbb{C} \to C$ defined by $t \mapsto (t^2, t^3)$.

§2. Projective Varieties

Projective Space. We define *n-dimensional projective space* to be the set

$$\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\sim,$$

where $\sim$ is the equivalence relation on $\mathbb{C}^{n+1} - \{0\}$ given by

$$(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n) \iff \text{there is } \lambda \in \mathbb{C}^* \text{ with } (a_0, \ldots, a_n) = \lambda(b_0, \ldots, b_n). \tag{2.1}$$

Here, we use $\mathbb{C}^*$ to denote $\mathbb{C} - \{0\}$, which is a group under multiplication. As we vary $\lambda \in \mathbb{C}^*$, the points $\lambda(b_0, \ldots, b_n)$ lie on a line through the origin. Thus we get a bijection

$$\mathbb{P}^n \simeq \{\text{lines through the origin in } \mathbb{C}^{n+1}\}.$$

Exercise 2.1. $\mathbb{P}^n$ contains the subset $(\mathbb{C}^*)^{n+1}/\sim$. Note also that $(\mathbb{C}^*)^{n+1}$ is a group under component-wise multiplication.

a. Show that on $(\mathbb{C}^*)^{n+1}$, the equivalence classes of $\sim$ are the cosets of the subgroup $H = \{(\lambda, \ldots, \lambda) \mid \lambda \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^{n+1}$. Conclude that $(\mathbb{C}^*)^{n+1}/H \subset \mathbb{P}^n$.
b. Construct a group isomorphism $(\mathbb{C}^*)^{n+1}/H \simeq (\mathbb{C}^*)^n$.

Exercise 2.1 shows that $\mathbb{P}^n$ contains an isomorphic copy of $(\mathbb{C}^*)^n$. $\mathbb{P}^n$ is a classic example of a toric variety.

We note that $\mathbb{P}^n$ has a *classical topology* inherited from the usual topology on $\mathbb{C}^{n+1} - \{0\}$.

Exercise 2.2. Let S^{2n+1} be the unit $(2n+1)$-sphere centered at the origin in $\mathbb{C}^{n+1}$.

a. Show that the natural map $S^{2n+1} \to \mathbb{P}^n$ is onto and conclude that $\mathbb{P}^n$ is compact.
b. Show that the fibers of $S^{2n+1} \to \mathbb{P}^n$ are isomorphic to S^1. This is the *Hopf fibration*.

Homogeneous Coordinates. A point p of $\mathbb{P}^n$ will be written $(a_0, \ldots, a_n)$. This is only unique up to the equivalence relation (2.1). We call $(a_0, \ldots, a_n)$ *homogeneous coordinates* of p. In some books, this is written $p = [a_0, \ldots, a_n]$ or $p = (a_0 : \ldots : a_n)$ to emphasize the non-unique nature of these coordinates. We prefer to write $p = (a_0, \ldots, a_n)$, where it will be clear from the context that we are using homogeneous coordinates.

Projective Varieties and Homogeneous Ideals. A polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ is *homogeneous of degree d* if every term of f has total degree d. This is equivalent to the identity

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n). \tag{2.2}$$

Exercise 2.3. Show that any $f \in \mathbb{C}[x_0, \ldots, x_n]$ can be written uniquely in the form $f = \sum_{d \geq 0} f_d$ where f_d is homogeneous of degree d. We call f_d the *homogeneous components* of f.

Now suppose that $f \in \mathbb{C}[x_0, \dots, x_n]$ is homogeneous of degree d. Given $p \in \mathbb{P}^n$, we can't define "$f(p)$" since using $p = (a_0, \dots, a_n)$ would give

$$f(p) = f(a_0, \dots, a_n),$$

while using $p = \lambda(a_0, \dots, a_n)$ would give

$$f(p) = f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f(a_0, \dots, a_n).$$

However, the equation $f(p) = 0$ is well-defined since $\lambda \in \mathbb{C}^*$. Thus, homogeneous polynomials $f_1, \dots, f_s \in \mathbb{C}[x_0, \dots, x_n]$ define the *projective variety*

$$\mathbf{V}(f_1, \dots, f_s) = \{a \in \mathbb{P}^n \mid f_1(a) = \cdots = f_s(a) = 0\} \subset \mathbb{P}^n.$$

To formulate this in terms of ideals, we say that an ideal $I \subset \mathbb{C}[x_0, \dots, x_n]$ is *homogeneous* if it is generated by homogeneous polynomials.

Exercise 2.4. Show that an ideal $I \subset \mathbb{C}[x_0, \dots, x_n]$ is homogeneous if and only if for all $f \in \mathbb{C}[x_0, \dots, x_n]$, we have $f \in I \Leftrightarrow I$ contains the homogeneous components of f.

If $I \subset \mathbb{C}[x_0, \dots, x_n]$ is a homogeneous ideal, then we have the projective variety

$$\mathbf{V}(I) = \{a \in \mathbb{C}^n \mid f(a) = 0 \text{ for all } f \in I\}.$$

Conversely, given a projective variety $V \subset \mathbb{C}^n$, we get the homogeneous ideal

$$\mathbf{I}(V) = \{f \in \mathbb{C}[x_0, \dots, x_n] \mid f(a) = 0 \text{ for all } a \in V\}.$$

Exercise 2.5. We call $\langle x_0, \dots, x_n\rangle \subset \mathbb{C}[x_0, \dots, x_n]$ the *irrelevant ideal*. Show that $\mathbf{V}(I) = \emptyset$ whenever I contains a power of the irrelevant ideal.

Exercise 2.5 is actually part of the projective version of the Nullstellensatz, which goes as follows. We refer the reader to [3, Chapter 8] for a proof.

Projective Nullstellensatz. *Let $I \subset \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous ideal.*
 a. *$\mathbf{V}(I) = \emptyset$ if and only if $\langle x_0, \dots, x_n\rangle^m \subset I$ for some $m \geq 0$.*
 b. *$\mathbf{V}(I) \neq \emptyset$ implies $\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}$.*

Most of the concepts defined for affine varieties in $\mathbb{C}^n$ can be extended to projective varieties in $\mathbb{P}^n$ in the obvious way:

- $W \subset V$ is a *subvariety* of a projective variety $V \subset \mathbb{P}^n$ if W is a projective variety in $\mathbb{P}^n$.
- If $V \subset \mathbb{P}^n$ is a projective variety, then we call $\mathbb{P}^n - V$ a *Zariski open* subset of $\mathbb{P}^n$.
- The *Zariski topology* is the topology on $\mathbb{P}^n$ whose open sets are the Zariski open sets.
- The *Zariski closure* $\overline{S}$ of a subset $S \subset \mathbb{P}^n$ is the smallest projective variety containing S.

Rational Functions on Projective Space. We've already seen that a homogeneous polynomial in $\mathbb{C}[x_1, \dots, x_n]$ does not give a function on $\mathbb{P}^n$. However, the quotient of two such polynomials works, provided they have the same degree. More precisely, suppose that $f, g \in \mathbb{C}[x_1, \dots, x_n]$ have degree d and that $g \neq 0$. Then (2.2) shows that we get a well-defined function

$$\frac{f}{g} : \mathbb{P}^n - \mathbf{V}(g) \longrightarrow \mathbb{C}$$

As in §1, we write this as $f/g : \mathbb{P}^n \dashrightarrow \mathbb{C}$ and say that f/g is a *rational function* on $\mathbb{P}^n$.

Exercise 2.6. The set of all rational functions on $\mathbb{P}^n$ is

$$\mathbb{C}(\mathbb{P}^n) = \Big\{\frac{f}{g} \mid f, g \in \mathbb{C}[x_0, \dots, x_n] \text{ homogeneous of equal degree, } g \neq 0\Big\}.$$

Prove that $\mathbb{C}(\mathbb{P}^n)$ is subfield of $\mathbb{C}(x_0, \dots, x_n)$.

Mappings Between Projective Varieties. Suppose that $V \subset \mathbb{P}^n$ is a projective variety and $f_0, \dots, f_m \in \mathbb{C}[x_0, \dots, x_n]$ are homogeneous polynomials all of the same degree. Then we say that $f_0, \dots, f_m$ have no *base points* on V if $V \cap \mathbf{V}(f_0, \dots, f_m) = \emptyset$.

Exercise 2.7. Suppose that $f_0, \dots, f_m \in \mathbb{C}[x_0, \dots, x_n]$ are homogeneous of degree d and have no base points on V. Prove that the map $(a_0, \dots, a_n) \mapsto (f_0(a_0, \dots, a_n), \dots, f_m(a_0, \dots, a_n))$ induces a well-defined function $F : V \longrightarrow \mathbb{P}^m$.

An important fact is that in the situation of Exercise 2.7, the image $F(V) \subset \mathbb{P}^m$ is a projective subvariety. When $V = \mathbb{P}^n$, this is proved in [3, Chapter 8], and the proof extends easily to cover the general case.

Exercise 2.8. When $V \subset \mathbb{C}^n$ is an affine variety and $F : V \to \mathbb{C}^m$ is a polynomial map, the image $F(V) \subset \mathbb{C}^m$ need not be a subvariety. For example, suppose that $V = \mathbf{V}(xy - 1) \subset \mathbb{C}^2$ and $F : V \to \mathbb{C}$ is $F(x, y) = x$. Prove that $F(V)$ is not a subvariety of $\mathbb{C}$. The fact that $F(V)$ is a subvariety in the projective case is one reason why projective varieties are so useful in algebraic geometry.

Affine Open Subsets. We can regard $\mathbb{P}^n$ as a union of affine spaces as follows. For $0 \leq i \leq n$, consider the Zariski open set $U_i = \mathbb{P}^n - \mathbf{V}(x_i)$.

Exercise 2.9. As above, $U_i = \mathbb{P}^n - \mathbf{V}(x_i) = \{(a_0, \dots, a_n) \in \mathbb{P}^n \mid a_i \neq 0\}$.
a. Show that $U_i \simeq \mathbb{C}^n$ via $(a_0, \dots, a_n) \mapsto (a_0/a_i, \dots, a_{i-1}/a_i, a_{i+1}/a_i, \dots, a_n/a_i)$.
b. Show that $\mathbf{V}(x_i) \simeq \mathbb{P}^{n-1}$ via $(a_0, \dots, a_n) \mapsto (a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$.
c. Show that $\mathbb{P}^n = U_0 \cup \dots \cup U_n$.

This exercise shows that we can regard $\mathbb{P}^n$ as $\mathbb{C}^n$ together with a copy of $\mathbb{P}^{n-1}$ "at infinity". Also, the open cover of Exercise 2.9 shows that projective varieties are unions of affine varieties.

Exercise 2.10. Let $V = \mathbf{V}(f_1, \dots, f_s) \subset \mathbb{P}^n$ be a projective variety. Prove that under the map $U_i \simeq \mathbb{C}^n$ from Exercise 2.9, $V \cap U_i$ corresponds to an affine variety defined by the vanishing of $\tilde{f}_j$, where $\tilde{f}_j(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = f_j(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$. We call $\tilde{f}_j$ the *dehomogenization* of f_j with respect to x_i.

Another way to think about $U_i \simeq \mathbb{C}^n$ is to use $x_0/x_i, \dots, x_{i-1}/x_i, x_{i-1}/x_i, \dots, x_n/x_i$ as variables on $\mathbb{C}^n$. Then the dehomogenization map of Exercise 2.10 is just $f \mapsto f/x_i^d$, where $f \in \mathbb{C}[x_0, \dots, x_n]$ is homogeneous of degree d. This approach preserves rational functions.

Exercise 2.11. Show that the map $f/g \mapsto (f/x_i^d)/(g/x_i^d)$ induces an isomorphism of fields

$$\mathbb{C}(\mathbb{P}^n) \simeq \mathbb{C}(x_0/x_i, \dots, x_{i-1}/x_i, x_{i+1}/x_i, \dots, x_n/x_i),$$

where $\mathbb{C}(\mathbb{P}^n)$ is the field of rational functions defined in Exercise 2.6.

Weighted Projective Space. We next discuss a generalization of $\mathbb{P}^n$. Given positive integers $q_0, \ldots, q_n$ satisfying $\gcd(q_0, \ldots, q_n) = 1$, we get the *weighted projective space*

$$\mathbb{P}(q_0, \ldots, q_n) = (\mathbb{C}^{n+1} - \{0\})/\!\sim,$$

where $\sim$ is the equivalence relation on $\mathbb{C}^{n+1} - \{0\}$ given by

$$(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n) \iff \text{there is } \lambda \in \mathbb{C}^* \text{ with } (a_0, \ldots, a_n) = (\lambda^{q_0} b_0, \ldots, \lambda^{q_n} b_n).$$

Obviously $\mathbb{P}(1, \ldots, 1) = \mathbb{P}^n$. We will eventually show that $\mathbb{P}(q_0, \ldots, q_n)$ is a toric variety. The following exercise shows that $\mathbb{P}(q_0, \ldots, q_n)$ contains a copy of $(\mathbb{C}^*)^n$.

Exercise 2.12. As above, let $q_0, \ldots, q_n$ be positive integers with $\gcd(q_0, \ldots, q_n) = 1$.
a. Prove that $(\mathbb{C}^*)^{n+1}/\widetilde{H} \subset \mathbb{P}(q_0, \ldots, q_n)$, where $\widetilde{H} = \{(\lambda^{q_0}, \ldots, \lambda^{q_n}) \mid \lambda \in \mathbb{C}^*\}$.
b. Prove that $(\mathbb{C}^*)^{n+1}/\widetilde{H} \simeq (\mathbb{C}^*)^n$. Hint: Make $(q_0, \ldots, q_n)$ the first column of a matrix $M \in \mathrm{GL}(n+1, \mathbb{Z})$ and use M to define an automorphism of $(\mathbb{C}^*)^{n+1}$.

We call $q_0, \ldots, q_n$ the *weights* of the weighted projective space. In terms of the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$, this means that x_i has degree q_i, and $f \in \mathbb{C}[x_0, \ldots, x_n]$ is *weighted homogeneous* of (weighted) degree d if

$$f(\lambda^{q_0} x_0, \ldots, \lambda^{q_n} x_n) = \lambda^d f(x_0, \ldots, x_n). \tag{2.3}$$

It is then easy to see that we can define *weighted projective subvarieties* in $\mathbb{P}(q_0, \ldots, q_n)$ using weighted homogeneous polynomials.

There are several ways to think about weighted projective spaces. The following two exercises give two ways to represent $\mathbb{P}(1,1,2)$.

Exercise 2.13. Consider $\mathbb{P}(1,1,2)$ with variables x_0, x_1, x_2 of degrees $1, 1, 2$ repsectively.
a. Show that $x_0^2, x_0x_1, x_1^2, x_2$ are (weighted) homogeneous of degree 2.
b. Show that $(a_0, a_1, a_2) \mapsto (a_0^2, a_0a_1, a_1^2, a_2)$ is a well-defined map $F : \mathbb{P}(1,1,2) \to \mathbb{P}^3$.
c. Show that the map F of part b is injective and that its image is the surface defined by the equation $y_0y_2 - y_1^2 = 0$ (where y_0, y_1, y_2, y_3 are the coordinates of $\mathbb{P}^3$).

Exercise 2.14. Show that $(a_0, b_0, c_0) \to (a_0, b_0, c_0^2)$ gives a well-defined map $\mathbb{P}^2 \to \mathbb{P}(1,1,2)$. Also show that this map is surjective and is two-to-one except above $(0,0,1) \in \mathbb{P}(1,1,2)$.

We can also cover a weighted projective space by affine open subsets, though in this case, the open sets will be affine varieties instead of affine space $\mathbb{C}^n$. Rather than work this out in general, we will restrict to the case of $\mathbb{P}(1,1,2)$. Here, we have the Zariski open sets $U_i = \{(a_0, a_1, a_2) \in \mathbb{P}(1,1,2) \mid a_i \neq 0\}$.

Exercise 2.15. Let U_0, U_1, U_2 be the subsets of $\mathbb{P}(1,1,2)$ defined above.
a. Show that $U_0 \simeq \mathbb{C}^2$ via $(a,b,c) \mapsto (b/a, c/a^2)$ and $U_1 \simeq \mathbb{C}^2$ via $(a,b,c) \mapsto (a/b, c/b^2)$.
b. Let $V = \mathbf{V}(xz - y^2) \subset \mathbb{C}^3$. Show that $U_2 \simeq V$ via $(a,b,c) \mapsto (a^2/c, ab/c, b^2/c)$.

One shows that $\mathbb{P}(1,1,2)$ is the abstract variety obtained by "gluing" two copies of $\mathbb{C}^2$ together with the affine variety U_2 from part c of Exercise 2.15. But we must first understand what "gluing" means.

§3. Abstract Varieties

The Definition of Manifold. To better understand the definition of abstract variety, we begin by recalling the definition of a C^∞ n-manifold. Such a manifold consists of a second countable Hausdorff topological space M together with an open cover U_α and homeomorphisms $f_\alpha : U_\alpha \to V_\alpha$, where $V_\alpha \subset \mathbb{R}^n$ is open, such that for every α, β, the composition

$$f_\beta \circ f_\alpha^{-1} : f_a(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism.

It turns out that there is a simpler, though more sophisticated, way of giving this definition. We begin with an open set $V \subset \mathbb{R}^n$. The *sheaf* of C^∞ functions on V, denoted $\mathcal{O}_V^\infty$, is defined by assigning to each open set $U \subset V$ the $\mathbb{R}$-algebra

$$\mathcal{O}_V^\infty(U) = \{f : U \to \mathbb{R} \mid f \text{ is a } C^\infty \text{ function}\}. \tag{3.1}$$

More generally, given a topological space X, we say that $\mathcal{F}$ is *sheaf of* $\mathbb{R}$*-algebras* on X if for every open set $U \subset X$, there is an $\mathbb{R}$-algebra $\mathcal{F}(U)$ such that:

- If $V \subset U$ are open, then there is an $\mathbb{R}$-algebra homomorphism $r_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$.*
- r_U^U is the identity and if $W \subset V \subset U$ are open, then $r_W^V \circ r_V^U = r_W^U$.*
- If $U = \bigcup_\alpha U_\alpha$, where U_α is open, then we have an exact sequence

$$0 \to \mathcal{F}(U) \to \prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta),$$

  where the second arrow is the map $\mathcal{F}(U) \to \mathcal{F}(U_\alpha)$ and the double arrows are the maps $\mathcal{F}(U_\alpha) \to \mathcal{F}(U_\alpha \cap U_\beta)$ and $\mathcal{F}(U_\beta) \to \mathcal{F}(U_\alpha \cap U_\beta)$.

Elements of $\mathcal{F}(U)$ are called *sections* of $\mathcal{F}$ over U, and when $V \subset U$, $r_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ the *restriction map*. In the third bullet, exactness at $\mathcal{F}(U)$ means that sections of $\mathcal{F}(U)$ are determined locally, i.e., two sections are equal if and only if their restrictions to the U_α are equal. Exactness at $\prod_\alpha \mathcal{F}(U_\alpha)$ means that compatible sections patch, i.e., sections over the U_α which agree on their intersections come from a section over U.

Given a sheaf $\mathcal{F}$ and $U \subset X$ open, the sections of $\mathcal{F}$ over U can be denoted

$$\mathcal{F}(U) = \Gamma(U, \mathcal{F}) = H^0(U, \mathcal{F}).$$

We will use the first of these in this section but will switch to the second for §9. When thinking in terms of sheaf cohomology, one usually uses the third.

When $\mathcal{F}$ is a sheaf of $\mathbb{R}$-algebras on X, we call the pair $(X, \mathcal{F})$ a *ringed space over* $\mathbb{R}$. For example, when $V \subset \mathbb{R}^n$ is open, (3.1) gives a ringed space over $\mathbb{R}$ denoted $(V, \mathcal{O}_V^\infty)$.

Exercise 3.1. Complete the following definitions:

a. The *restriction* $\mathcal{F}\big|_U$ of a sheaf $\mathcal{F}$ on X to an open set $U \subset X$ is defined by ...
b. Ringed spaces $(X, \mathcal{F})$, $(Y, \mathcal{G})$ over $\mathbb{R}$ are *isomorphic* if there are a homeomorphism $\phi : X \to Y$ and, for $U \subset Y$ open, an $\mathbb{R}$-algebra isomorphism $\phi_U^\# : \mathcal{G}(U) \to \mathcal{F}(\phi^{-1}(U))$, such that ...

We can now reformulate the definition of C^∞ n-manifold.

* These two bullets say that $\mathcal{F}$: **Open sets of** $X \to \mathbb{R}$**-Algebras** is a contravariant functor.

Exercise 3.2. Let n be a positive integer and let $(M, \mathcal{O}_M)$ be a ringed space over $\mathbb{R}$. Assume that every point in M has a neighborhood U such that $(U, \mathcal{O}_M\big|_U)$ is isomorphic to $(V, \mathcal{O}_V^\infty)$ for some open subset $V \subset \mathbb{R}^n$. Prove that M has the structure of a C^∞ n-manifold.

Exercise 3.3. Conversely, let M be a C^∞ n-manifold as defined at the beginning of the section.

a. Given $U \subset M$ open, define what it means for $f : U \to \mathbb{R}$ to be C^∞.
b. Use the definition given in part a to define the sheaf $\mathcal{O}_M$ of C^∞ functions on M and show that $(M, \mathcal{O}_M)$ is a ringed space over $\mathbb{R}$ which satisfies the condition of Exercise 2.2.

For a C^∞ n-manifold M, the sheaf $\mathcal{O}_M$ of Exercise 3.3 is called the *structure sheaf* of M.

Exercise 3.4. Give sheaf-theoretic definitions of a C^k n-manifold and a complex n-manifold.

The Structure Sheaf of an Affine Variety. We first show that some of the constructions for $\mathbb{C}^n$ given in §1 generalize to an arbitrary affine variety $V = \operatorname{Spec}(R)$.

Exercise 3.5. Let $V = \operatorname{Spec}(R)$ be an affine variety.

a. Given an ideal $I \subset R$, define $\mathbf{V}(I) \subset V$. Then prove that $\mathbf{V}(I)$ is a subvariety of V and that all subvarieties of V arise in this way.
b. Given a subvariety $W \subset V$, define $\mathbf{I}(W) \subset R$ and prove that $\mathbf{I}(W)$ is a radical ideal of R.
c. Prove the Nullstellensatz, i.e., that $\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}$ for any ideal $I \subset R$.
d. Prove the Hilbert Basis Theorem, i.e., that any ideal $I \subset R$ can be written in the form $I = \langle f_1, \ldots, f_s \rangle$, where $f_1, \ldots, f_s \in R$.

In §1, we defined the Zariski open $V_f \subset V$ for any $f \in R$. Furthermore, when V is irreducible, we showed that $V_f = \operatorname{Spec}(R_f)$, where

$$R_f = \{g/f^m \in \mathbb{C}(V) \mid g \in R,\ m \geq 0\}$$

is the localization of R at f, as defined in Exercise 1.9 of §1.

Exercise 3.6. Show that the sets V_f form a basis of the Zariski topology of V.

The *structure sheaf* of an irreducible affine variety $V = \operatorname{Spec}(R)$ is the sheaf of $\mathbb{C}$-algebras in the Zariski topology defined as follows. Given a Zariski open $U \subset V$, a function $\phi : U \to \mathbb{C}$ is *regular* if for every $p \in V$, there is $f_p \in R$ such that $p \in V_{f_p} \subset U$ and $\phi\big|_{V_{f_p}} \in R_{f_p}$. Then

$$\mathcal{O}_V(U) = \{\phi : U \to \mathbb{C} \mid \phi \text{ is a regular function}\}.$$

We will not show that $\mathcal{O}_V$ is a sheaf of $\mathbb{C}$-algebras—we refer the reader to [4] or [5] for the details of the proof. These references also show how to define $\mathcal{O}_V$ when V is not irreducible.

Exercise 3.7. Let $V = \mathbb{C}^2$ and set $U = \mathbb{C}^2 - \{(0,0)\}$. Show that $\mathcal{O}_V(U) = \mathbb{C}[x, y]$.

The structure sheaf $\mathcal{O}_V$ has two important properties.

Theorem 3.1. *Let $V = \mathrm{Spec}(R)$ be an irreducible affine variety.*
a. $\mathcal{O}_V(V) = R$.
b. *If $f \in R$, then* $\mathcal{O}_V\big|_{V_f} = \mathcal{O}_{V_f}$.

Proof. For part a, it suffices to show $\mathcal{O}_V(V) \subset R$. If $\phi : V \to \mathbb{C}$ is a morphism, then for each $p \in V$, there are $f_p, g_p \in R$ such that $\phi = g_p/f_p^{m_p}$ and $f_p(p) \neq 0$. Let $I = \langle f_p^{m_p} \mid p \in V\rangle \subset R$. It follows easily that $\mathbf{V}(I) = \emptyset$, so that by the Nullstellensatz, $\sqrt{I} = \mathbf{I}(\mathbf{V}(I)) = \mathbf{I}(\emptyset) = R$. Thus $1 \in R$, which implies that $1 = \sum_{p\in S} h_p f_p^{m_p}$ where $h_p \in R$ and $S \subset V$ is finite. Then

$$\phi = \sum_{p\in S} h_p f_p^{m_p}\, \phi = \sum_{p\in S} h_p\, g_p \in R.$$

For part b, let $U \subset V$ be Zariski open. If $\phi : U \to \mathbb{C}$ is a morphism, then for every $p \in U$, there is $p \in V_{f_p} \subset U$ such that $\phi \in R_{f_p}$. Now suppose in addition that $U \subset V_f$. If we regard f_p as an element of R_f, then one easily sees that

$$\phi \in R_{f_p} \subset R_{ff_p} = (R_f)_{f_p}.$$

Furthermore, $V_{f_p} \cap V_f = (V_f)_{f_p}$ shows that $p \in (V_f)_{f_p} \subset U \subset V_f$. By definition, this implies that $\phi \in \mathcal{O}_{V_f}(U)$, and part b now follows easily. □

Combining parts a and b of Theorem 3.1, we conclude that

$$\mathcal{O}_V(V_f) = \mathcal{O}_V\big|_{V_f}(V_f) = \mathcal{O}_{V_f}(V_f) = R_f$$

when $V = \mathrm{Spec}(R)$ and $f \in R$.

The Definition of Abstract Variety. We now give the main definition of this section.

Definition 3.2. *An* **abstract variety** $(X, \mathcal{O}_X)$ *is a ringed space over $\mathbb{C}$ where each $p \in X$ has a neighborhood U such that the restriction $(U, \mathcal{O}_X\big|_U)$ is isomorphic (as a ringed space over $\mathbb{C}$) to $(V, \mathcal{O}_V)$ for some affine variety V.*

Given an abstract variety $(X, \mathcal{O}_X)$, an open set $U \subset X$ is an *affine open* if $(U, \mathcal{O}_X\big|_U)$ is isomorphic (over $\mathbb{C}$) to the ringed space of an affine variety. The topology on X is called the *Zariski topology* since it restricts to the Zariski topology in each affine open subset.

Exercise 3.8. Let $(X, \mathcal{O}_X)$ be an abstract variety and let $U \subset X$ be Zariski open. Show that every section $\phi \in \mathcal{O}_X(U)$ gives a function $\phi : U \to \mathbb{C}$. We say that ϕ a *regular function* on U.

Exercise 3.9. Let $(X, \mathcal{O}_X)$ be an abstract variety and let $Y \subset X$ be Zariski closed. If $U_1 \subset Y$ is open, define $\mathcal{O}_Y(U_1)$ to be the set of all functions $\phi : U_1 \to \mathbb{C}$ such that for every $p \in U_1$, there is $U \subset X$ open and a regular function $\tilde{\phi} : U \to \mathbb{C}$ with $p \in U \cap Y \subset U_1$ and $\phi\big|_{U\cap Y} = \tilde{\phi}\big|_{U\cap Y}$.
a. Show that $U_1 \mapsto \mathcal{O}_Y(U_1)$ is a sheaf of $\mathbb{C}$-algebras on Y.
b. When $(X, \mathcal{O}_X)$ is an affine variety and $Y \subset X$ is a subvariety, prove that the sheaf defined in part a is precisely the sheaf of regular functions on Y
c. When $(X, \mathcal{O}_X)$ is an abstract variety and $Y \subset X$ is Zariski closed, prove that the sheaf of part a makes $(Y, \mathcal{O}_Y)$ into an abstract variety.

Given an abstract variety $(X, \mathcal{O}_X)$, we say that $Y \subset X$ is a *subvariety* if it is Zariski closed. The above exercise shows that Y inherits the structure of an abstract variety in a natural way. We define $(X, \mathcal{O}_X)$ to be *irreducible* if X is not the union of two proper subvarieties.

Exercise 3.10. Prove that an abstract variety is irreducible if and only if it is connected and every affine open subset is irreducible.

Let us show that $\mathbb{P}^n$ can be regarded as an abstract variety. In order to define the structure sheaf $\mathcal{O}_{\mathbb{P}^n}$, we will use the field of rational functions $\mathbb{C}(\mathbb{P}^n)$ defined in Exercise 2.6 of §2. If $U \subset \mathbb{P}^n$ is Zariski open, then a function $\phi : U \to \mathbb{C}$ is *regular* if for each $p \in U$, there is $f/g \in \mathbb{C}(\mathbb{P}^n)$ such that $g(p) \neq 0$ and $\phi\big|_{U \setminus U \cap \mathbf{V}(g)} = (f/g)\big|_{U \setminus U \cap \mathbf{V}(g)}$. Then

$$\mathcal{O}_{\mathbb{P}^n}(U) = \{\phi : U \to \mathbb{C} \mid \phi \text{ is a regular function}\}$$

defines a sheaf $\mathcal{O}_{\mathbb{P}^n}$ on $\mathbb{P}^n$. We also have the affine open sets $U_i = \{(a_0, \dots, a_n) \in \mathbb{P}^n \mid a_i \neq 0\}$. In Exercise 2.11, we noted that if we regard $x_0/x_i, \dots, x_{i-1}/x_i, x_{i-1}/x_i, \dots, x_n/x_i$ as coordinates on $\mathbb{C}^n$, then the map $U_i \simeq \mathbb{C}^n$ induces an isomorphism

$$\mathbb{C}(\mathbb{P}^n) \simeq \mathbb{C}(x_0/x_i, \dots, x_{i-1}/x_i, x_{i-1}/x_i, \dots, x_n/x_i).$$

Using this isomorphism, it is easy to see that $(U_i, \mathcal{O}_{\mathbb{P}^n}\big|_{U_i})$ is isomorphic to $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$. This proves that $\mathbb{P}^n$ is an abstract variety. By Exercise 3.10, we see that $\mathbb{P}^n$ is irreducible.

As is customary, we often write an abstract variety $(X, \mathcal{O}_X)$ as simply X, and we will also drop the "abstract". Thus, "the variety X" is short for "the abstract variety $(X, \mathcal{O}_X)$".

Finally, a variety X also has a *classical topology*, which is the coarsest topology on X that agrees with the classical topology on every affine open subset of X. The structure sheaf $\mathcal{O}_X$ is not a sheaf in the classical topology. However, one can define the closely related sheaf $\mathcal{O}_X^{\mathrm{an}}$ of *analytic functions* on X, which is a sheaf in the classical topology. We call $(X, \mathcal{O}_X^{\mathrm{an}})$ the *complex analytic space* associated to the variety X. See [5, Appendix B] and [9] for more details.

The Function Field of an Irreducible Variety. If X is irreducible, then a *rational function* on X is a regular function $\phi : U \to \mathbb{C}$, where U is a nonempty Zariski open. Two rational functions are *equivalent* if they agree on some nonempty Zariski open, and the set of equivalence classes is denoted $\mathbb{C}(X)$. One can prove that $\mathbb{C}(X)$ is a field, called the *function field* of X.

Exercise 3.11. Let U be an affine open of an irreducible variety X. Prove that $\mathbb{C}(U) \simeq \mathbb{C}(X)$.

We say that $\phi \in \mathbb{C}(X)$ is *defined* at $p \in X$ there is a regular function $\phi' : U \to \mathbb{C}$ such that $p \in U$ and ϕ is equivalent to ϕ'. For $\phi \in \mathbb{C}(X)$, the set $\{p \in X \mid \phi \text{ is defined at } p\}$ is the largest Zariski open on which ϕ is defined.

The Local Ring of a Point. Given an irreducible variety X and a point $p \in X$, we define the *local ring* of X at p to be

$$\mathcal{O}_{X,p} = \{\phi \in \mathbb{C}(X) \mid \phi \text{ is defined at } p\}.$$

The key feature of $\mathcal{O}_{X,p}$ is described in the following exercise.

Exercise 3.12. Let $\mathcal{O}_{X,p}$ defined as above.

a. Show that $\mathfrak{m}_{X,p} = \{\phi \in \mathcal{O}_{X,p} \mid \phi(p) = 0\}$ is a maximal ideal of $\mathcal{O}_{X,p}$.
b. Given $\phi \in \mathcal{O}_{X,p}$, show that $\phi(p) \neq 0$ implies that $\phi^{-1} \in \mathcal{O}_{X,p}$.
c. Use part b to show that $\mathfrak{m}_{X,p}$ is the unique maximal ideal of $\mathcal{O}_{X,p}$.

In general, a commutative ring R with unit is called a *local ring* if it has a unique maximal ideal $\mathfrak{m}$. Thus $\mathcal{O}_{X,p}$ is a local ring by Exercise 3.12.

For an arbitrary variety X (not necessarily irreducible), one can define the local ring $\mathcal{O}_{X,p}$ to be the direct limit

$$\mathcal{O}_{X,p} = \varinjlim_{p \in U} \mathcal{O}_X(U),$$

where the limit is over all neighborhoods U of p. This is described in [4] and [5].

Morphisms. A *morphism* or *regular map* consists of a continuous map $\phi : X \to Y$ and, for each Zariski open $U \subset Y$, a $\mathbb{C}$-algebra homomorphism $\phi^{\#} : \mathcal{O}_Y(U) \to \mathcal{O}_X(\phi^{-1}(U))$, such that:

- $\phi^{\#}$ is compatible with restriction maps.
- For each $p \in X$, the map of local rings $\phi^{\#}_p : \mathcal{O}_{Y,\phi(p)} \to \mathcal{O}_{X,p}$ induced by $\phi^{\#}$ is a *local homomorphism*, meaning that $\mathfrak{m}_{Y,\phi(p)} = (\phi^{\#}_p)^{-1}(\mathfrak{m}_{X,p})$.

A morphism $(\phi, \phi^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is usually written $\phi : X \to Y$.

An important results is that if $V = \operatorname{Spec}(R)$ and $W = \operatorname{Spec}(S)$ are irreducible affine varieties, then giving a morphism $\phi : V \to W$ is equivalent to giving a $\mathbb{C}$-algebra homomorphism $\phi^* : S \to R$. This is proved in [4] and [5].

Gluing Together Affine Varieties. We first observe that a variety X can be constructed by "gluing together" affine varieties along Zariski open subsets. Namely, Definition 3.2 implies that X has an affine open cover U_α, so that $f_\alpha : U_\alpha \simeq V_\alpha$, where V_α is an affine variety. Then, for any α, β, the set

$$V_{\alpha\beta} = f_\alpha(U_\alpha \cap U_\beta) \subset V_\alpha$$

is Zariski open in V_α, and the map

$$g_{\alpha\beta} = f_\beta \circ f_\alpha^{-1} : V_{\alpha\beta} \to V_{\beta\alpha}$$

is an isomorphism of Zariski open subsets. Furthermore, these maps are compatible as follows:

- $g_{\alpha\alpha} = 1_{V_\alpha}$ for every α.
- $g_{\beta\gamma}\big|_{V_{\beta\alpha} \cap V_{\beta\gamma}} \circ g_{\alpha\beta}\big|_{V_{\alpha\beta} \cap V_{\alpha\gamma}} = g_{\alpha\gamma}\big|_{V_{\alpha\beta} \cap V_{\alpha\gamma}}$ for every α, β, γ.

We call these the *compatibility conditions*.

Conversely, suppose we have a collection

$$(\{V_\alpha\}_\alpha, \{V_{\alpha\beta}\}_{\alpha,\beta}, \{g_{\alpha\beta}\}_{\alpha,\beta}),$$

where each V_α is an affine variety, $V_{\alpha\beta} \subset V_\alpha$ is Zariski open, and the $g_{\alpha\beta} : V_{\alpha\beta} \simeq V_{\beta\alpha}$ are isomorphisms which satisfy the above compatibility conditions. Then we get the topological space

$$X = \coprod_\alpha V_\alpha / \sim$$

where $a \in V_\alpha$ is equivalent to $b \in V_\beta$ if $a \in V_{\alpha\beta}$ and $b = g_{\alpha\beta}(a)$. Furthermore, the structure sheaves $\mathcal{O}_{V_\alpha}$ patch to give a sheaf $\mathcal{O}_X$, and from here it is straightforward to prove that $(X, \mathcal{O}_X)$ is a variety with an affine open cover U_α such that $U_\alpha \simeq V_\alpha$ for every α. In this situation, we say that X is obtained from the V_α by *gluing* them together along the $V_{\alpha\beta}$ via the $g_{\alpha\beta}$.

Exercise 3.13. Let $V_0 = V_1 = \mathbb{C}$, $V_{01} = V_{10} = \mathbb{C} - \{0\}$ and $g_{01}(x) = g_{10}(x) = x^{-1}$. Prove that this data determines the variety $\mathbb{P}^1$.

Cartesian Products. If X and Y are varieties, then their cartesian product $X \times Y$ exists, though the definition is subtle. The difficulty is that the usual product topology on $X \times Y$ gives the wrong topology. Here is an example.

Exercise 3.14. Show that the product topology on $\mathbb{C} \times \mathbb{C}$, where we use the Zariski topology on each factor, is not the Zariski topology on $\mathbb{C}^2$.

To get the correct definition of cartesian product, we begin with affine varieties. Suppose that $V = \mathbf{V}(f_1, \ldots, f_s) \subset \mathbb{C}^n$, with variables $x_1, \ldots, x_n$ and $W = \mathbf{V}(g_1, \ldots, g_t) \subset \mathbb{C}^m$, with variables $y_1, \ldots, y_m$. Also let R and S be the coordinate rings of V and W respectively.

Exercise 3.15. Let $V = \mathrm{Spec}(R)$ and $W = \mathrm{Spec}(S)$ be as above.
a. Show that $V \times W \subset \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^{n+m}$ is the affine variety $\mathbf{V}(f_1, \ldots, f_s, g_1, \ldots, g_t)$, where $f_i(x_1, \ldots, x_n), g_j(y_1, \ldots, y_m) \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]$.
b. Show that the coordinate ring of $V \times W$ is $R \otimes_{\mathbb{C}} S$.

Thus the cartesian product of $V = \mathrm{Spec}(R)$ and $W = \mathrm{Spec}(S)$ is $V \times W = \mathrm{Spec}(R \otimes_{\mathbb{C}} S)$.

In the general case, we think of X as obtained by gluing together Zariski open subsets of U_α, and similarly Y comes from gluing together Zariski open subsets of U'_β. Then $X \times Y$ is constructed by gluing together the affine varieties $U_\alpha \times U'_\beta$ along suitable Zariski open subsets. We omit the details of the construction, which can be found in [4] and [5].

As an example, $\mathbb{P}^n \times \mathbb{P}^m$ can be constructed by this method. If $x_0, \ldots, x_n$ are coordinates on $\mathbb{P}^n$ and $y_0, \ldots, y_m$ are coordinates on $\mathbb{P}^m$, then one can show that $V \subset \mathbb{P}^n \times \mathbb{P}^m$ is Zariski closed if and only if $V = \mathbf{V}(f_1, \ldots, f_s)$, where $f_i \in \mathbb{C}[x_0, \ldots, x_n; y_0, \ldots, y_m]$ is *bihomogeneous*, meaning that it is separately homogeneous in the x_i and in the y_j.

Finally, we should mention that if X and Y are varieties, then the classical topology on $X \times Y$ is the product of the classical topologies on X and Y respectively.

§4. Separated, Quasi-Compact, Complete, and Normal Varieties

Separatedness. Given any variety X, the *diagonal map* of X is the map $\Delta : X \to X \times X$ defined by $\Delta(p) = (p, p)$ for $p \in X$. Then X is *separated* if the image of the diagonal map $\Delta(X)$ is Zariski closed in $X \times X$. Here are some examples of separated and non-separated varieties.

Exercise 4.1. Consider the variety X constructed by identifying two copies of $\mathbb{C}$ along $\mathbb{C} - \{0\}$ (in the language of §3, this corresponds to $V_0 = V_1 = \mathbb{C}$, $V_{01} = V_{10} = \mathbb{C} - \{0\}$ and $g_{01}(x) = g_{10}(x) = x$). Show that X is not separated.

Exercise 4.2. Prove that $\mathbb{C}^n$ is separated by considering $\mathbf{V}(x_1 - y_1, \ldots, x_n - y_n) \subset \mathbb{C}^n \times \mathbb{C}^n$.

Exercise 4.3. Prove that a subvariety of a separated variety is separated.

Combining Exercises 4.2 and 4.3, we see that affine varieties are always separated. We will omit the proof that $\mathbb{P}^n$ is separated. By Exercise 4.3, it follows that every projective variety is separated.

In [9], Serre characterized separatedness in terms of the classical topology as follows.

Theorem 4.1. *A variety X is separated if and only if it is Hausdorff in the classical topology.*

For example, this theorem makes it easy to see that the variety X of Exercise 4.1 is not Hausdorff (the two copies of the origin in X do not have disjoint neighborhoods). This also gives a quick proof of Exercise 4.2 since $\mathbb{C}^n$ is Hausdorff in the classical topology.

Here are some properties of separated varieties.

Proposition 4.2. *Let X be a separated variety.*

a. *If U and V are affine open subsets of X, then $U \cap V$ is also an affine open of X.*
b. *If $f, g : Y \to X$ is a morphism of varieties, then $\{y \in Y \mid f(y) = g(y)\}$ is a subvariety of Y.*

Proof. For part a, we note that $\Delta : X \to X \times X$ identifies $U \cap V$ with $\Delta(X) \cap (U \times V)$. We know that $\Delta(X)$ is Zariski closed in $X \times X$ by the definition of separated. It follows that $\Delta(X) \cap (U \times V)$ is Zariski closed in $U \times V$. But $U \times V$ is affine by Exercise 3.15, which implies that $\Delta(X) \cap (U \times V) \simeq U \cap V$ is also affine.

To prove part b, consider $\phi : Y \to X \times X$ defined by $\phi(y) = (f(y), g(y))$. Since

$$\{y \in Y \mid f(y) = g(y)\} = \phi^{-1}(\Delta(X)),$$

we see that $\{y \in Y \mid f(y) = g(y)\}$ is Zariski closed in Y since ϕ is continuous in the Zariski topology and (by separatedness) $\Delta(X)$ is Zariski closed in $X \times X$. □

When studying differentiable manifolds, one always assumes that the underlying topological space is Hausdorff. Similarly, in algebraic geometry, the varieties of interest are almost always separated. For this reason, we henceforth reserve the term *variety* for a separated variety. A non-separated variety will be called a *pre-variety*.

Quasi-Compactness. We say that a variety X is *quasi-compact* if X is the union of finitely many affine open subsets. Any affine variety is quasi-compact. An more interesting example is $\mathbb{P}^n$, which is quasi-compact since it the union of the affine open subsets $U_i = \mathbb{P}^n - \mathbf{V}(x_i)$.

Here are the main properties of quasi-compact varieties.

Proposition 4.3. *Let X be a quasi-compact variety. Then:*

a. *Every subvariety of X is quasi-compact.*
b. *Every Zariski open subset of X is quasi-compact.*
c. *Every Zariski open cover of X has a finite subcover.*

Proof. Suppose that $X = U_1 \cup \cdots \cup U_r$ where U_i is an affine open subset of X. Then part a is obvious since $Y = (Y \cap U_1) \cup \cdots \cup (Y \cap U_r)$ where $Y \cap U_i$ is an affine open of Y.

For part b, $U = (U \cap U_1) \cup \cdots \cup (U \cap U_r)$ shows that we can assume that $X = \mathrm{Spec}(R)$ is affine. Then $X - U = \mathrm{V}(I)$ for some ideal $I \subset R$. The Hilbert basis theorem implies $I = \langle f_1, \ldots, f_s \rangle$, and it follows that $U = \bigcup_{i=1}^s X_{f_i}$.

Finally, we leave part c as Exercise 4.4 below. □

Exercise 4.4. Prove that every Zariski open cover of a quasi-compact variety has a finite subcover.

This exercise explains there the term "quasi-compact" comes from. As with separatedness, the varieties of interest to algebraic geometers are almost always quasi-compact. Hence, from now on, whenever we say *variety*, we will mean a separated quasi-compact abstract variety.

Completeness and Properness. A variety X is *complete* if the following conditions hold:

- X is separated.

- X is quasi-compact.
- For every other variety Y, the projection map $X \times Y \to Y$ is closed, meaning that the projection of a Zariski closed set in $X \times Y$ is Zariski closed in Y.

Exercise 4.5. Prove that a subvariety of a complete variety is complete.

Exercise 4.6. Prove that the cartesian product of two complete varieties is complete.

Exercise 4.7. Prove that a variety X is complete if and only if for all $m \geq 1$, the projection map $X \times \mathbb{C}^m \to \mathbb{C}^m$ is closed.

The most basic example of a complete variety is $\mathbb{P}^n$.

Theorem 4.4. *$\mathbb{P}^n$ is complete.*

For a proof, note that Theorem 6 of [3, Chapter 8, §5] implies that the projection map $\mathbb{P}^n \times \mathbb{C}^m \to \mathbb{C}^m$ is closed. By Exercise 4.7, it follows that $\mathbb{P}^n$ is complete, and then any projective variety is complete by Exercise 4.5.

The completeness of $\mathbb{P}^n$ is closely related to elimination theory. To see why, suppose we have polynomials

$$f_1, \ldots, f_s \in \mathbb{C}[x_0, \ldots, x_n, y_1, \ldots, y_m]$$

which are homogeneous in $x_0, \ldots, x_n$. We can think of the f_i as homogeneneous polynomials in the x_i whose coefficients depend on the "parameters" y_j. Question: For which values of the parameters y_j do the equations

$$f_1 = \cdots = f_s = 0 \tag{4.1}$$

have a nontrivial solution in the x_i?

To answer this question, observe that (4.1) defines a subvariety $W \subset \mathbb{P}^n \times \mathbb{C}^m$, and the values of the y_j for which (4.1) has a nontrivial solution is the image of W under the projection map $\mathbb{P}^n \times \mathbb{C}^m \to \mathbb{C}^m$. Since $\mathbb{P}^n$ is complete, this image is a variety in $\mathbb{C}^m$. In other words, there are polynomials $g_1, \ldots, g_l \in \mathbb{C}[y_1, \ldots, y_m]$ such that (4.1) has a nontrivial solution for the parameter values $y_j = b_j$ if and only if

$$g_1(b_1, \ldots, b_m) = \cdots = g_l(b_1, \ldots, b_m) = 0. \tag{4.2}$$

Chapter 8 of [3] gives an algorithm for finding the polynomials g_i. We say that (4.2) is obtained from (4.1) by *eliminating* the variables $x_0, \ldots, x_n$. This is *projective elimination theory*.

Serre's paper [9] characterizes completeness in terms of the classical topology as follows.

Theorem 4.5. *A variety X is complete if and only if it is compact in the classical topology.*

This theorem and the Hopf fibration $S^{2n+1} \to \mathbb{P}^n$ give another proof that $\mathbb{P}^n$ is complete (see Exercise 2.2). In algebraic geometry, completeness is a very useful property. Here is a result which indicates some reasons why.

Theorem 4.6. *Let X be a complete variety.*

a. *If $\phi : X \to Y$ is a morphism, then its image $\phi(X)$ is a subvariety of Y.*
b. *If $\phi : \mathbb{C}^* \to X$ is a morphism, then ϕ extends uniquely to a morphism $\tilde{\phi} : \mathbb{C} \to X$.*
c. *X is affine if and only if X is a finite set of points.*

d. If X is connected, then every morphism $\phi : X \to \mathbb{C}$ is constant.

Part b of this theorem says that if X is complete and $\phi : \mathbb{C}^* \to X$ is a morphsim, then $\lim_{t\to 0} \phi(t)$ exists as a unique element of X whenever X is complete (in fact, $\lim_{t\to\infty} \phi(t)$ also exists, so that ϕ extends to a map $\mathbb{P}^1 \to X$). Part c says that affine varieties are very far from being complete since these concepts coincide only for finite sets of points.

Finally, completeness closely related to the idea of a *proper morphism* $\phi : X \to Y$. We will not give the definition here (we would need to define *fibered products* and *morphisms of finite type*). The reader should consult [4] and [5] for the full definition. Turning to Serre's classic paper [9] yet again, we can characterize properness in terms of the classical topology as follows.

Theorem 4.7. *A morphism $\phi : X \to Y$ is proper if and only if it is proper in the classical topology, meaning that $\phi^{-1}(C) \subset X$ is compact whenever $C \subset Y$ is compact.*

Exercise 4.8. Prove that X is complete if and only if $X \to \{\text{pt}\}$ is proper, where $\{\text{pt}\}$ is the variety consisting of a single point.

Normality. A variety X is *normal* if it is irreducible and the local ring $\mathcal{O}_{X,p}$ is integrally closed for every $p \in X$. In order to relate this to the definition of normal affine variety given in §1, we will need the following exercise.

Exercise 4.9. Let R be an integral domain with field of fractions K. A subset $S \subset R$ is a *multiplicative subset* if $1 \in S$, $0 \notin S$, and S is closed under multiplication. Then the *localization* of R at S is $R_S = \{a/b \in K \mid a \in R, b \in S\}$.

a. Prove that R_S is the smallest subring of K containing R such that every $s \in S$ is invertible in R_S.
b. Prove that if R is integrally closed, then R_S is integrally closed.

We now show that for affine varieties, our two notions of normal coincide.

Proposition 4.8. *Let $V = \operatorname{Spec}(R)$ be an irreducible affine variety. The R is integrally closed if and only if the local ring $\mathcal{O}_{V,p}$ is integrally closed for all $p \in V$.*

Proof. We know from §1 that $p \in V$ corresponds to a maximal ideal $M \subset R$. Then $R - M$ is a multiplicative subset since maximal implies prime, and one easily shows that

$$\mathcal{O}_{V,p} = R_{R-M}.$$

If R is integrally closed, then Exercise 4.9 implies that $\mathcal{O}_{V,p}$ is also integrally closed. Conversely, if all of the $\mathcal{O}_{V,p}$ are integrally closed, then one easily shows that

$$\bigcap_{p\in V} \mathcal{O}_{V,p}$$

is also integrally closed. However, this intersection is precisely $\mathcal{O}_V(V)$, which equals R by part a of Theorem 3.1 of §3. It follows that R is integrally closed. $\square$

Proposition 4.8 has a following immediate corollary.

Corollary 4.9. *An irreducible variety X is normal if and only if it is a union of affine varieties $U_\alpha = \operatorname{Spec}(R_\alpha)$ where R_α is integrally closed.*

§5. Smooth and Quasismooth Varieties

The Dimension of a Variety. We were using a very naive notion of *dimension* when we asserted that $\mathbb{C}^n$ and $(\mathbb{C}^*)^n$ have dimension n. For an arbitrary variety X, there are several ways to define $\dim X$ rigorously. In the irreducible case, we do this as follows.

Definition 5.1. *The* **dimension** *of an irreducible variety X is:*
- *The transcendence degree of $\mathbb{C}(X)$ (this is the maximal number of algebraically independent elements of $\mathbb{C}(X)$).*
- *The maximum number n such that one can find distinct irreducible subvarieties*

$$\emptyset \neq V_0 \subset V_1 \subset \cdots \subset V_n = X.$$

It is not at all obvious that these definitions coincide, but they do—see [5]. In the affine or projective case, one can also define dimension using the degree of an appropriate Hilbert polynomial. This approach is used in [3].

Some important results concerning the dimension of a variety are:
- $\mathbb{C}^n$, $(\mathbb{C}^*)^n$ and $\mathbb{P}^n$ have dimension n.
- The dimension of a variety is the maximum of the dimensions of its irreducible components.
- If W is a subvariety of V, then $\dim W \leq \dim V$. Furthermore, if V is irreducible and W is a proper subvariety, then $\dim W < \dim V$.
- $\dim X \times Y = \dim X + \dim Y$.
- If V is an irreducible affine variety and $f \in \mathbb{C}[V]$ is not invertible, then every irreducible component of $\mathbf{V}(f) \subset V$ has codimension 1.
- Let $V \subset \mathbb{P}^n$ be irreducible of positive dimension and pick $f \in \mathbb{C}[x_0, \ldots, x_n]$. If f doesn't vanish on V, then then every irreducible component of $V \cap \mathbf{V}(f) \subset V$ has codimension 1.

The Dimension of a Variety at a Point. The dimension of a variety X at a point $p \in X$, denoted $\dim_p X$, is defined as either:
- The maximum of the dimensions of the irreducible components of X which contain p.
- The Krull dimension of the local ring $\mathcal{O}_{X,p}$ (this is one less than the maximum length of a chain of prime ideals in $\mathcal{O}_{X,p}$).

Some of the properties of the dimension at a point include:
- $\dim X = \max_{p\in X} \dim_p X$.
- If $p \in Y \subset X$, then $\dim_p Y \leq \dim_p X$.
- If $p \in X$ and $q \in Y$, then $\dim_{(p,q)} X \times Y = \dim_p X + \dim_p Y$.

The Zariski Tangent Space. In multivariable calculus, one defines the tangent space at a point of a surface in $\mathbb{R}^3$, and this generalizes to the tangent space at a point of a differentiable manifold. In algebraic geometry, the *Zariski tangent space* plays a similar role.

Definition 5.2. *Let p be a point of a variety X and let $\mathfrak{m}_{X,p}$ be the maximal ideal of the local ring $\mathcal{O}_{X,p}$. Then the* **Zariski tangent space** *is defined to be*

$$T_p(X) = \operatorname{Hom}_{\mathbb{C}}(\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2, \mathbb{C}).$$

Exercise 5.2. Use $\mathcal{O}_{X,p}/\mathfrak{m}_{X,p} \simeq \mathbb{C}$ to prove that $\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2$ has a natural structure as a vector space over $\mathbb{C}$. This shows that Definition 5.2 makes sense.

Exercise 5.3. Let $V \subset \mathbb{C}^n$ be an affine variety and $p = (a_1, \ldots, a_n) \in V$.

a. Show that $\mathfrak{m}_{\mathbb{C}^n,p} = \langle x_1 - a_1, \ldots, x_n - a_n\rangle \subset \mathcal{O}_{\mathbb{C}^n,p}$.
b. Show that $\mathfrak{m}_{\mathbb{C}^n,p}/\mathfrak{m}_{\mathbb{C}^n,p}^2$ has dimension n, and conclude that $\dim_{\mathbb{C}} T_p(\mathbb{C}^n) = n$.
c. Use the surjection $\mathcal{O}_{\mathbb{C}^n,p} \to \mathcal{O}_{V,p}$ to construct natural inclusion $T_p(V) \subset T_p(\mathbb{C}^n)$.
d. Conclude that $\dim_{\mathbb{C}} T_p(V) \leq n$.

In the affine case, page 32 of [5] shows how to compute the Zariski tangent spaces as follows.

Lemma 5.3. *Let $V \subset \mathbb{C}^n$ be a affine variety and let $p \in V$. Also assume that $\mathbf{I}(V) = \langle f_1, \ldots, f_s\rangle$. For each i, let*

$$d_p(f_i) = \frac{\partial f_i}{\partial x_1}(p)\, x_1 + \cdots + \frac{\partial f_i}{\partial x_n}(p)\, x_n.$$

The $T_p(V)$ is isomorphic to the subspace of $\mathbb{C}^n$ defined by $d_p(f_1) = \cdots = d_p(f_s) = 0$.

Exercise 5.4. Let $V = \mathbf{V}(x^3 - y^2) \subset \mathbb{C}^2$. For each $p \in V$, show that $\dim_{\mathbb{C}} T_p(V) = 1$ unless p is the origin, in which case the dimension is 2.

Exercise 5.5. If $p \in X$ and $q \in Y$, prove that $T_{(p,q)}(X \times Y) \simeq T_p(X) \oplus T_q(Y)$. Hint: Reduce to the affine case and use Lemma 2.3. See also Exercise 3.15 of §3.

In general, we always have $\dim_{\mathbb{C}} T_p(X) \geq \dim_p X$. See Exercise 5.10 of [5, Chapter I].

Smooth Varieties. As with dimension, there are many ways to define smoothness.

Definition 5.4. *A variety X is* **smooth** *or* **nonsingular** *at $p \in X$ if $\dim_{\mathbb{C}} T_p(X) = \dim_p(X)$. We say that p is a* **singular** *point of X if it is not a smooth point.*

Since $T_p(X) = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2, \mathbb{C})$, we see that X is smooth at p when $\dim_p(X)$ equals the dimension of $\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2$ as a vector space over $\mathcal{O}_{X,p}/\mathfrak{m}_{X,p}$. In terms of commutative algebra, this means that $p \in X$ is smooth if and only if $\mathcal{O}_{X,p}$ is a regular local ring.

By Exericse 5.3, every point of $\mathbb{C}^n$ is smooth (such a variety is called *smooth*). For a point of a subvariety of $\mathbb{C}^n$, we can test for smoothness as follows.

Exercise 5.6. Let $V \subset \mathbb{C}^n$ be an affine variety and let $\mathbf{I}(V) = \langle f_1, \ldots, f_s\rangle$. Also let $p \in V$ and set $d = \dim_p V$. Then prove that V is smooth at p if and only if the Jacobian matrix

$$J_p(f_1, \ldots, f_s) = \Big(\frac{\partial f_i}{\partial x_j}(p)\Big)_{1 \leq i \leq s, 1 \leq j \leq n}$$

has rank $n - d$. Hint: Use Lemma 5.3.

Exercise 5.7. Let $V = \mathbf{V}(xy - zw)$. Prove that the origin is the only singular point of V.

Exercise 5.8. Let $p \in X$ and $q \in Y$. Prove that $X \times Y$ is smooth at (p, q) if and only if X and Y are smooth at p and q respectively. Hint: Use Exercise 5.5 and the properties of $\dim_p X$.

Exercise 5.9. Given a variety X, the set $X_{\mathrm{sing}} = \{p \in X \mid p \text{ is singular}\}$ is the *singular locus* of X. Use Exercise 5.5 to prove that X_{sing} is a subvariety of X. (With more work, one can show that X_{sing} is a proper subvariety of X. See Theorem 5.3 of [5].)

Local Analytic Equivalence. When we want to say that two varieties are *locally* the same, we have to be careful to specify what we mean by "local".

Suppose that we have $p \in X$ and $q \in Y$, where X and Y are varieties. Then, in the Zariski topology, X and Y being "locally equivalent" at p and q respectively means that there are Zariski open sets $p \in U \subset X$ and $q \in V \subset Y$ such that $U \simeq V$ as varieties. Since Zariski open subsets are huge, this notion of "local equivalence" is not very useful.

Exercise 5.10. Show that $p \in X$ and $q \in Y$ are "locally equivalent" in the above sense if and only if the local rings $\mathcal{O}_{X,p}$ and $\mathcal{O}_{Y,q}$ are isomorphic as $\mathbb{C}$-algebras.

In §3, we discussed the sheaf $\mathcal{O}_X^{\mathrm{an}}$ of analytic functions on X, which is a sheaf in the classical topology. This allows one to define an analytic (or holomorphic) map between classical open sets in varieties. Then X and Y are *analytically equivalent* at p and q if there are classical open sets $p \in U \subset X$ and $q \in V \subset Y$ such that $U \simeq V$ as analytic varieties. Here are two nice facts about local analytic equivalence:

- As in Exercise 5.7, X and Y are analytically equivalent at p and q if and only if there is a $\mathbb{C}$-algebra isomorphism between the local rings $\mathcal{O}_{X,p}^{\mathrm{an}}$ and $\mathcal{O}_{Y,q}^{\mathrm{an}}$.
- $p \in X$ is smooth if and only if it is analytically equivalent to $0 \in \mathbb{C}^n$.

Finite Quotients of Affine Space. Let G be a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$. Then G acts on $\mathbb{C}^n$, and the quotient $\mathbb{C}^n/G$ is the set of G-orbits. By Chapter 7 of [3], we can turn this set into an affine variety as follows.

Proposition 5.5. *Given a finite subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$, let $\mathbb{C}[x_1, \ldots, x_n]^G \subset \mathbb{C}[x_1, \ldots, x_n]$ be the subring of invariant polynomials. There is a natural bijection $\mathbb{C}^n/G \simeq \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_n]^G)$.*

Understanding the structure of $\mathbb{C}[x_1, \ldots, x_n]^G$ is one of the goals of *invariant theory*. In some cases, the quotient $\mathbb{C}^n/G$ is still smooth.

Exercise 5.11. Let $C_m \in \mathrm{GL}(n, \mathbb{C})$ be the matrix with $e^{2\pi i/m}, 1, \ldots, 1$ on the main diagonal and 0's elsewhere, and let $G = \{C_m^i \mid 0 \leq i \leq m-1\}$. Use the map $\mathbb{C}^n \to \mathbb{C}^n$ given by $(a_1, a_2, \ldots, a_n) \mapsto (a_1^m, a_2, \ldots, a_n)$ to prove that $\mathbb{C}^n/G \simeq \mathbb{C}^n$. Also, what is $\mathbb{C}[x_1, \ldots, x_n]^G$?

Exercise 5.12. Let the symmetric group S_n be embedded in $\mathrm{GL}(n, \mathbb{C})$ as the set of permutation matrices. Then S_n acts on $\mathbb{C}^n$ by permuting coordinates. Prove that $\mathbb{C}^n/S_n \simeq \mathbb{C}^n$. Hint: Elementary symmetric polynomials.

A matrix in $\mathrm{GL}(n, \mathbb{C})$ is a *complex reflection* if it is conjugate to the matrix C_m of Exercise 5.7, and $G \subset \mathrm{GL}(n, \mathbb{C})$ is a *complex rotation group* if it is generated by complex rotations. The Shephard-Todd-Chevalley theorem says that $\mathbb{C}^n/G \simeq \mathbb{C}^n$ if and only if G is a complex reflection group. A proof can be found in [10, Section 2.4].

Exercise 5.13. Show that the $n \times n$ permutation matrices form a complex reflection group.

Exercise 5.14. Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite subgroup and let H be subgroup of G generated by the elements of G which are complex reflections. Prove that H is normal in G.

We next define a special type of finite matrix group.

Definition 5.6. *A finite subgroup $G \subset \mathrm{GL}(n,\mathbb{C})$ is* **small** *if it contains no complex reflections other than the identity.*

Small subgroups were introduced by Prill in order to obtain a one-to-one correspondence between groups and quotients. More precisely, we have the following results, proved in [7]:

- If $G \subset \mathrm{GL}(n,\mathbb{C})$ is finite, then in a classical neighborhood of the origin, $\mathbb{C}^n/G$ is analytically equivalent to the quotient of $\mathbb{C}^n$ by a small subgroup. (The rough idea is that if H is the subgroup of Exericise 5.14, then $\mathbb{C}^n/H \simeq \mathbb{C}^n$ by the Shephard-Todd-Chevalley Theorem, and in a classical neighborhood of the origin, the action of G/H on $\mathbb{C}^n$ is analytically equivalent to the linear action of a small subgroup.)
- If G_1 and G_2 are small subgroups of $\mathrm{GL}(n,\mathbb{C})$ which give analytically equivalent singularities, then G_1 and G_2 are conjugate in $\mathrm{GL}(n,\mathbb{C})$.

Quasimooth Varieties. We now define a type of singularity which is close to being smooth.

Definition 5.7. *A point p of a variety X is a* **finite quotient singularity** *if there is a small subgroup $G \subset \mathrm{GL}(n,\mathbb{C})$ such that $p \in X$ is analytically equivalent to $0 \in \mathbb{C}^n/G$. Then X is* **quasismooth** *or has* **finite quotient singularities** *or is* **$\mathbb{Q}$-smooth** *if every point of p is a finite quotient singularity.*

Note that the definition of finite quotient singularity allows G to be the trivial subgroup of $\mathrm{GL}(n,\mathbb{C})$. It follows that any smooth variety is quasismooth. Here is an example to show that the converse is not true.

Exercise 5.15. Let $V = \mathbf{V}(xz - y^2) \subset \mathbb{C}^3$.

a. Show that the origin is the unique singular point of V.
b. Let $G = \{\pm I\} \subset \mathrm{GL}(2,\mathbb{C})$. If we think of $\mathbb{C}^2$ as $\mathrm{Spec}(\mathbb{C}[a,b])$, then show that $\mathbb{C}[a,b]^G = \mathbb{C}[a^2, ab, b^2]$.
c. Show that $\mathbb{C}[a^2,ab,b^2] \simeq \mathbb{C}[x,y,z]/\langle xz - y^2\rangle$, and conclude that $\mathbb{C}^2/G \simeq V$.

For the surface $V \subset \mathbb{C}^3$ of this exercise, $0 \in V$ is not smooth by part a and is a finite quotient singularity by part c. Since all other points of V are smooth, we see that V is quasismooth but not smooth.

We can generalize Exercise 5.15 as follows.

Proposition 5.8. *Let $G \subset \mathbb{C}^n$ be a small subgroup. Then $\mathbb{C}^n/G$ is quasismooth.*

Proof. The definition of quasismooth guarantees that $0 \in \mathbb{C}^n/G$ is a finite quotient singularity. But what about the other points of $\mathbb{C}^n/G$? Given $v \in \mathbb{C}^n$, let $G_v = \{g \in G \mid g \cdot v = v\}$ be its isotropy subgroup. We will show that $v \in \mathbb{C}^n/G$ is analytically equivalent to $0 \in \mathbb{C}^n/G_v$.

First observe that $w \mapsto w + v$ is equivariant with respect to the action of G_v, as is $w \mapsto w - v$. This gives an isomorphism of varieties $\mathbb{C}^n/G_v \to \mathbb{C}^n/G_v$ which takes 0 to v. Thus $0 \in \mathbb{C}^n/G_v$ is analytically equivalent to $0 \in \mathbb{C}^n/G_v$.

Hence we need only show that $v \in \mathbb{C}^n/G_v$ is analytically equivalent to $v \in \mathbb{C}^n/G$. Let $\{g_i\}$ be left coset representatives for G/G_v. Then $\mathbb{C}^n/G$ is obtained from $\mathbb{C}^n/G_v$ by identifying $w \in \mathbb{C}^n/G_v$ with $g_i \cdot w$ for all i. Since the points $g_i \cdot v$ are distinct in $\mathbb{C}^n/G_v$, we can find a classical neighborhood U of $v \in \mathbb{C}^n/G_v$ such that the neighborhoods $g_i \cdot U$ are disjoint. The g_i act on $\mathbb{C}^n/G_v$ as isomorphisms of varieties, which implies that $v \in U \subset \mathbb{C}^n/G_v$ is analytically equivalent to a neighborhood of $v \in \mathbb{C}^n/G$. This gives the desired analytic equivalence. □

Exercise 5.16. Prove that a cartesian product of quasismooth varieties is quasismooth.

§6. The Local Ring of an Irreducible Hypersurface

Let X be an irreducible variety with function field $\mathbb{C}(X)$. A subvariety $Y \subset X$ is a *hypersurface* if every irreducible component of Y has codimension 1 in X.

The Local Ring. Let $Y \subset X$ be an irreducible hypersurface. Then consider the set

$$\mathcal{O}_{X,Y} = \{f \in \mathbb{C}(X) \mid f \text{ is defined on a nonempty Zariski open subset of } Y\}.$$

To understand this, recall that $f \in \mathbb{C}(X)$ means that there is a nonempty Zariski open $Y \subset X$ and $f : U \to \mathbb{C}$ is a morphism. Then $f \in \mathcal{O}_{X,Y}$ when we can find such a U satisfying $U \cap Y \neq 0$.

Exercise 6.1. Prove that $\mathcal{O}_{X,Y}$ is a local ring and that the maximal ideal consists of those $f \in \mathcal{O}_{X,Y}$ which vanish on Y.

Exercise 6.2. Let $Y = \mathbf{V}(x) \subset \mathbb{C}^2$.

a. Prove that
$$\mathcal{O}_{\mathbb{C}^2,Y} = \Big\{\frac{P(x,y)}{Q(x,y)} \mid P(x,y), Q(x,y) \in \mathbb{C}[x,y],\ Q(0,y) \neq 0\Big\}.$$

b. Given $f \in \mathbb{C}(x,y)$, prove that $f = x^m g$, where $m \in \mathbb{Z}$ and $g \in \mathcal{O}_{\mathbb{C}^2,Y}$ is a unit. Hint: Write $f = P/Q$, where $Q(0,y) \neq 0$. Explain why $P = x^k P'$ and $Q = x^l Q'$, where $P(0,y)$ and $Q(0,y)$ are nonzero.

c. Prove that every nonzero ideal of $\mathcal{O}_{\mathbb{C}^2,Y}$ is of the form $\langle x^m\rangle$ for some $m \geq 0$.

Given $f \in \mathbb{C}(x,y)$, Exercise 6.2 tells us that $f = x^m g$ for $m \in \mathbb{Z}$ and g a unit in $\mathcal{O}_{\mathbb{C}^2,Y}$. We call m the *order of vanishing* of f on $Y = \mathbf{V}(x) \subset \mathbb{C}^2$ and denote it by $\mathrm{ord}_Y(f)$.

Discrete Valuation Rings. The crucial observation is that Exercise 6.2 generalizes to any normal variety. Let R be an integral domain with field of fractions K, and set $K^* = K - \{0\}$. Then R is a *discrete valuation ring* if there is a surjective function

$$\mathrm{ord}_R : K^* \to \mathbb{Z}$$

such that every for $a, b \in K^*$, we have:

- $\mathrm{ord}_R(ab) = \mathrm{ord}_R(a) + \mathrm{ord}_R(b)$.
- $\mathrm{ord}_R(a+b) \geq \min(\mathrm{ord}_R(a), \mathrm{ord}_R(b))$ provided $a + b \neq 0$.
- $R = \{a \in K^* \mid \mathrm{ord}_R(a) \geq 0\} \cup \{0\}$.

We say that ord_R is a *valuation* on K and that R is its *valuation ring*.

Exercise 6.3. Let R be a discrete valuation ring.

a, Prove that R is a local ring with $\mathfrak{m} = \{a \in R \mid \mathrm{ord}_R(a) > 0\}$ as maximal ideal.

b. Let $a \in R$ satisfy $\mathrm{ord}_R(a) = 1$ (a exists because ord_R is onto). Prove that $\mathfrak{m} = \langle a\rangle$.

c. Let a be as in part b. Prove that any nonzero ideal of R is of the form $\langle a^m\rangle$ for some $m \geq 0$.

Exercise 6.4. Prove that a discrete valuation ring is an integrally closed one-dimensional Noetherian local ring. (A ring R is *Noetherian* if every ideal of R is finitely generated, i.e., if the Hilbert Basis Theorem holds for R.)

Here are two classic examples of discrete valuation rings.

Exercise 6.5. Let p be prime. Prove that $\mathbb{Z}_{(p)} = \{a/b \mid a, b \in \mathbb{Z},\ \gcd(p,b) = 1\}$ is a discrete valuation ring. This gives the *p-adic valuation*, denoted ord_p.

Exercise 6.6. Let $\mathbb{C}\{\{z\}\}$ the ring of complex power series with positive radius of convergence. Prove that $\mathbb{C}\{\{z\}\}$ is a discrete valuation ring and that the valuation gives the order of vanishing of a nonzero element of $\mathbb{C}\{\{z\}\}$.

For us, the main result we need is as follows.

Theorem 6.1. *Let Y be an irreducible hypersurface in a normal variety X. Then $\mathcal{O}_{X,Y}$ is a discrete valuation ring.*

Proof. The argument requires substantial amounts of commutative algebra. We will omit the details and just sketch the ideas involved. One begins with the following observations:

- $\mathcal{O}_{X,Y}$ is integrally closed since the localization of a integrally closed domain is integrally closed.
- $\mathcal{O}_{X,Y}$ has dimension 1 as a ring since Y having codimension 1 in X.
- $\mathcal{O}_{X,Y}$ is Noetherian since the localization of a Noetherian ring is Noetherian.

Thus $\mathcal{O}_{X,Y}$ is a integrally closed one-dimensional Noetherian local ring. A classic result states that any such ring is a discrete valuation ring (and conversely, as you showed in Exercise 6.4). The commutative algebra used here can be found in [1], especially Chapter 9. □

In the situation of Theorem 6.1, the corresponding valuation is written

$$\mathrm{ord}_Y : \mathbb{C}(X)^* \to \mathbb{Z}.$$

Given $f \in \mathbb{C}(X)^*$, we say that f *vanishes to order m* along Y if $m = \mathrm{ord}_Y(f) > 0$ and *has a pole of order m* on Y if $m = -\mathrm{ord}_Y(f) > 0$.

§7. Weil Divisors on Normal Varieties

Weil Divisors. A *Weil divisor* on a normal variety X is a finite formal sum

$$D = \sum_{i=1}^{s} a_i D_i$$

where the D_i are distinct irreducible hypersurfaces of X and $a_i \in \mathbb{Z}$. The set of all Weil divisors is a group under addition and is denoted

$$\mathrm{WDiv}(X).$$

We say that $D = \sum_{i=1}^{s} a_i D_i$ is *effective* if $a_i \geq 0$ for all i, and we write this as

$$D \geq 0.$$

Note that any Weil divisor can be written uniquely as the difference of two effective Weil divisors.

The Divisor of a Rational Function. Given $f \in \mathbb{C}(X)$, we can define $\mathrm{ord}_Y(f)$ for every irreducible hypersurface $Y \subset X$. This gives a Weil divisor as follows.

Proposition 7.1. *Let X be normal and $f \in \mathbb{C}(X)$ be nonzero. Then there are at most finitely many hypersuraces $Y \subset X$ such that $\mathrm{ord}_Y(f) \neq 0$. Thus we can define the Weil divisor*

$$\mathrm{div}(f) = \textstyle\sum_Y \mathrm{ord}_Y(f)\, Y.$$

Proof. Let $U \subset \mathbb{C}$ be a Zariski open where $f : U \to \mathbb{C}$ is a nonzero morphism. Then $U' = U - f^{-1}(0)$ is also Zariski open in X. If $Y \subset X$ is an irreducible hypersurface with $Y \cap U' \neq \emptyset$, then $\mathrm{ord}_Y(f) = 0$ since f is defined but nonvanishing on $Y \cap U'$. Thus $\mathrm{ord}_Y(f) \neq 0$ implies $Y \subset X - U'$. Since $X - U'$ is a proper subvariety of X and Y has codimension 1, it follows that Y must be an irreducible component of $X - U'$. Then we are done since $X - U'$ has at most finitely many irreducible components. □

We sometimes write $\mathrm{div}(f) = \mathrm{div}_0(f) - \mathrm{div}_\infty(f)$, where

$$\begin{aligned}\mathrm{div}_0(f) &= \sum_{\mathrm{ord}_Y(f)>0} \mathrm{ord}_Y(f)\, Y\\ \mathrm{div}_\infty(f) &= \sum_{\mathrm{ord}_Y(f)<0} -\mathrm{ord}_Y(f)\, Y.\end{aligned}$$

We call $\mathrm{div}_0(f)$ (resp. $\mathrm{div}_\infty(f)$) the *divisor of zeros* of f (resp. the *divisor of poles* of f). Note that these are effective divisors.

Exercise 7.1. Explain why $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$ and $\mathrm{div}(1/f) = -\mathrm{div}(f)$ for $f, g \in \mathbb{C}(X)^*$.

Exercise 7.2. Let $f \in \mathbb{C}[t]$ be a polynomial of degree n, and write $f = c(x - a_1)^{m_1} \cdots (x - a_r)^{m_r}$, where $a_1, \ldots, a_r \in \mathbb{C}$ are distinct.
a. When $X = \mathbb{C}$, show that $\mathrm{div}(f) = \sum_{i=1}^r m_i \{a_i\}$.
b. When $X = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, show that $\mathrm{div}(f) = \sum_{i=1}^r m_i \{a_i\} - n\{\infty\}$.

Finally, we need to know when the divisor of a rational function vanishes.

Proposition 7.2. *Let X be a normal variety and let $f \in \mathbb{C}(X)^*$. Then $\mathrm{div}(f) \geq 0$ if and only if $f : X \to \mathbb{C}$ is a morphism, i.e., $f \in \mathcal{O}_X(X)$.*

Proof. If $f : X \to \mathbb{C}$ is a morphism, then $f \in \mathcal{O}_{X,Y}$ for every Y, which in turn implies $\mathrm{ord}_Y(f) \geq 0$. Hence $\mathrm{div}(f) \geq 0$. Going the other way, suppose that $\mathrm{div}(f) \geq 0$. Then

$$f \in \textstyle\bigcap_Y \mathcal{O}_{X,Y}, \tag{7.1}$$

where the intersection is over all irreducible hypersurfaces of X. Hence f is defined on a nonempty Zariski open subset of every irreducible hypersurface. It follows that f is defined outside of a subvariety of codimension at least 2. Since X is normal, a standard result in commutative algebra implies that f is defined everywhere (see Exercise 7.3 below). □

Exercise 7.3. Let $X = \mathrm{Spec}(R)$, where R is integrally closed. Let K be the fraction field of R and suppose that $f \in K$ has $\mathrm{div}(f) = 0$.
a. Show that (7.1) implies that $f \in \bigcap_{\mathfrak{p}} R_{\mathfrak{p}}$, where:
- The intersection is over all prime ideals $\mathfrak{p} \subset R$ such that $\mathbf{V}(\mathfrak{p})$ has codimension 1 in X (these are called prime ideal of *height* 1, written $\mathrm{ht}(\mathfrak{p}) = 1$).
- $R_{\mathfrak{p}}$ is the localization of R at the multiplicative subset $R - \mathfrak{p}$ (in Exercise 4.9 of §4, this was written $R_{R-\mathfrak{p}}$).

b. A theorem in commutative algebra states that $R = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} R_{\mathfrak{p}}$ whenever R is Noetherian and integrally closed. A proof can be found in [6, §12]. Explain how this completes the proof of Proposition 7.2.

While the proof of Proposition 7.2 uses a lot of commutative algebra, there is also some nice intuition coming from several complex variables. Suppose that $U \subset \mathbb{C}^2$ is a classical neighborhood of the origin and that f is holomorphic on $U - \{(0,0)\}$. Then *Hartogs' Lemma* asserts that f extends automatically to a holomorphic function on U. This applies more generally as follows: if X is a normal analytic space and f is homomorphic on $X - Y$, where Y has codimension at least 2, then f extends to a holomorphic function on X.

For a connected complete variety X, we learned in Theorem 4.6 of §4 that the only morphisms $X \to \mathbb{C}$ are constant. This gives the following corollary of Proposition 7.2.

Corollary 7.3. *Let X be a complete normal variety and let $f \in \mathbb{C}(X)^*$. Then $\mathrm{div}(f) \geq 0$ if and only if f is constant.*

Since $\mathrm{div}(1/f) = -\mathrm{div}(f)$ by Exercise 7.1, we see that $\mathrm{div}(f) = 0$ if and only if $\mathrm{div}(f) \geq 0$ and $\mathrm{div}(1/f) \geq 0$. Hence we get another corollary of Proposition 7.2.

Corollary 7.4. *Let X be a normal variety and let $f \in \mathbb{C}(X)^*$. Then $\mathrm{div}(f) = 0$ if and only if $f : X \to \mathbb{C}^*$ is a morphism, i.e., $f \in \mathcal{O}_X(X)^*$ (the group of invertible elements of $\mathcal{O}_X(X)$).*

Linearly Equivalent Divisors and the Divisor Class Group. As above, let X be a normal variety. We say that two Weil divisors $D_1, D_2 \in \mathrm{WDiv}(X)$ are *linearly equivalent*, written $D_1 \sim D_2$, if there is $f \in \mathbb{C}(X)^*$ such that $\mathrm{div}(f) = D_1 - D_2$. Furthermore, we say that $D \in \mathrm{WDiv}(X)$ is a *principal divisor* if $D \sim 0$, i.e., $D = \mathrm{div}(f)$ for some $f \in \mathbb{C}(X)$.

Exercise 7.4. Let $\sim$ be defined as above.

a. Use Exercise 2.1 to show that set of principal divisors is a subgroup of $\mathrm{WDiv}(X)$.
b. Show that $\sim$ is an equivalence relation on $\mathrm{Div}(X)$.

The subgroup of principal divisors is denoted $\mathrm{Div}_0(X)$ (§8 will explain this notation). Parts a and b of Exercise 7.4 are linked, of course, since $\sim$ is the equivalance relation coming from the subgroup $\mathrm{Div}_0(X)$. The quotient group

$$\mathrm{Cl}(X) = \mathrm{WDiv}(X)/\mathrm{Div}_0(X)$$

is the *divisor class group* of X. It consists of equivalence classes of linearly equivalent divisors. Given $D \in \mathrm{Div}(X)$, its divisor class in $\mathrm{Cl}(X)$ is denoted $[D]$.

Here is an important exact sequence involving the class group.

Exercise 7.5. Let X be a normal variety. Use Corollary 7.4 to prove that there is an exact sequence

$$1 \to \mathcal{O}_X(X)^* \to \mathbb{C}(X)^* \to \mathrm{WDiv}(X) \to \mathrm{Cl}(X) \to 0,$$

where the map $\mathbb{C}(X)^* \to \mathrm{WDiv}(X)$ is $f \mapsto \mathrm{div}(f)$ and $\mathrm{WDiv}(X) \to \mathrm{Cl}(X)$ is $D \mapsto [D]$.

One pretty result we will need is the following. A proof can be found in Proposition 6.2 of [5, II.6].

Theorem 7.5. *For a normal affine variety $X = \text{Spec}(R)$, the class group $\text{Cl}(X)$ is trivial if and only if R is a unique factorization domain.*

The class group $\text{Cl}(X)$ is sometimes denoted $A_{n-1}(X)$, where $n = \dim X$. More generally, one can define *Chow groups* for $A_k(X)$ for any irreducible variety X.

In another direction, let $\mathcal{O}_K$ be the ring of algebraic integers in a number field K. Then the scheme $X = \text{Spec}(R)$ is normal, and $\text{Cl}(X)$ can defined as above. One can show that in this case, $\text{Cl}(X)$ is the *ideal class group* of K as defined in algebraic number theory.

§8. Cartier Divisors on Normal Varieties

We will give a slightly non-standard treatment of Cartier divisors which works nicely on normal varieties.

Our Definition. Let $D = \sum_{i=1}^{s} a_i D_i$ be a Weil divisor on a normal variety X. If $U \subset X$ is a nonempty Zariski open subset, then the *restriction* of D to U is the is Weil divisor

$$D\big|_U = \sum_{U \cap D_i \neq \emptyset} a_i \, U \cap D_i.$$

We now define a special class of Weil divisors.

Definition 8.1. *Let D be a Weil divisor on a normal variety X.*

a. *D is* **locally principal** *if there is an open cover $\{U_i\}_{i\in I}$ of X such that $D\big|_{U_i}$ is principal for every $i \in I$.*
b. *D is* **Cartier** *if it is locally principal.*

A principal divisor is obviously locally principal. Thus $\text{div}(f)$ is Cartier for all $f \in \mathbb{C}(X)^*$.

Exercise 8.1. Let D and E be Cartier divisors. Prove that $D + E$ and $-D$ are Cartier.

Exercise 8.2. Let $D \sim E$ be linearly equivalent Weil divisors. Prove that D is Cartier if and only if E is Cartier.

If D is locally principal relative to the open cover $\{U_i\}_{i\in I}$, then we can find $f_i \in \mathbb{C}(X)^*$ such that $D\big|_{U_i} = \text{div}(f_i)$ on U_i. We say that $\{(U_i, f_i)\}_{i\in I}$ is *local data* for D.

Exercise 8.3. Let $\{(U_i, f_i)\}_{i\in I}$ be local data for a Cartier divisor D.

a. Prove that $f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)^*$ for all $i, j \in I$. Hint: Use Corollary 7.4.
b. Prove that D is effective if and only if $f_i \in \mathcal{O}_X(U_i)$ for all $i \in I$. Hint: Use Proposition 7.2.

For an example of a Weil divisor which is not Cartier, consider the affine surface $X = \mathbf{V}(xy - z^2) \subset \mathbb{C}^3$. The x-axis $Y = \mathbf{V}(y, z)$ is contained in V, so that Y is a Weil divisor on X. However, one can show that Y is *not* a Cartier divisor (see Example 6.11.3 in [5, II.6]).

There is one nice case where Weil and Cartier divisors coincide.

Theorem 8.2. *Let X be a normal variety such that the local ring $\mathcal{O}_{X,p}$ is a unique factorization domain for every $p \in X$. Then every Weil divisor on X is Cartier.*

This is proved in Proposition 6.11 of [5, II.6]). We should also mention that if X is smooth, then $\mathcal{O}_{X,p}$ is a unique factorization domain for all p. It follows that Weil and Cartier divisors coincide on smooth varieties.

The Standard Definition. Definition 8.1 differs from what one usually finds in the literature. The more common definition starts with local data $\{(U_i, f_i)\}_{i\in I}$ satisfying part a of Exercise 8.3 and defines two local data $\{(U_i, f_i)\}_{i\in I}$, $\{(V_j, g_j)\}_{j\in J}$ to be equivalent if $f_i/g_j \in \mathcal{O}_X(U_i \cap V_j)^*$ for all $(i,j) \in I \times J$. Then a *Cartier divisor* is an equivalance class of local data.

There is also more sophisticated way to define Cartier divisors which avoids equivalence classes. We have the sheaf $\mathcal{O}_X^*$ whose sections over U are the invertible elements in the ring $\mathcal{O}_X(U)$, and we can also regard $\mathbb{C}(X)^*$ as a constant sheaf on X. Then one can show that a Cartier divisor is a global section of the quotient sheaf $\mathbb{C}(X)^*/\mathcal{O}_X^*$. See [5, page 141] for details.

The Picard Group. We denote the set of all Cartier divisors on a normal variety X by

$$\mathrm{Div}(X).$$

This is a subgroup of $\mathrm{WDiv}(X)$ by Exercise 8.1. Furthermore, since every principal divisor is Cartier, we have $\mathrm{Div}_0(X) \subset \mathrm{Div}(X)$. Then we define the *Picard group* of X to be the quotient

$$\mathrm{Pic}(X) = \mathrm{Div}(X)/\mathrm{Div}_0(X). \tag{8.1}$$

We will give a more sophisticated definition of $\mathrm{Pic}(X)$ in §10. (Note that (8.1) explains why the group of principal divisors is denoted $\mathrm{Div}_0(X)$ rather than $\mathrm{WDiv}_0(X)$.) Since $\mathrm{Div}(X)$ is a subgroup of $\mathrm{WDiv}(X)$, we get a canonical injection

$$\mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X).$$

In analogy with Exercise 2.7, we have the following exact sequence.

Exercise 8.4. Let X be a normal variety. Prove that there is an exact sequence

$$1 \to \mathcal{O}_X(X)^* \to \mathbb{C}(X)^* \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0,$$

where the map $\mathbb{C}(X)^* \to \mathrm{Div}(X)$ sends f to $\mathrm{div}(f)$ and the map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ is the natural homomorphism.

§9. The Sheaf of a Weil Divisor

Definition and Basic Properties. Let D be a Weil divisor on a normal variety X. We will show that D determines a sheaf $\mathcal{O}_X(D)$ of $\mathcal{O}_X$-modules on X. As noted in §3, the sections of a sheaf $\mathcal{F}$ over $U \subset X$ can be written

$$\mathcal{F}(U) = \Gamma(U, \mathcal{F}) = H^0(U, \mathcal{F}).$$

For $\mathcal{O}_X(D)$, we will find it convenient to use the middle notation. Thus, given a Zariski open subset $U \subset X$, we define

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in \mathbb{C}(X)^* \mid (\mathrm{div}(f) + D)\big|_U \geq 0\} \cup \{0\}.$$

Lemma 9.1. *The above definition makes $\mathcal{O}_X(D)$ into a sheaf of $\mathcal{O}_X$-modules on X.*

Proof. We first show that $\Gamma(U, \mathcal{O}_X(D))$ is an additive subgroup of $\mathbb{C}(X)$. It suffices to prove this for $U = X$. Let $D = \sum_{i=1}^{s} a_i D_i$. Then $f \in \Gamma(X, \mathcal{O}_X(D))$ if and only if $\mathrm{ord}_{D_i}(f) \geq -a_i$ for all i. If $g \in \mathbb{C}(X)^*$ also has this property, then so does $f + g$ since

$$\mathrm{ord}_{D_i}(f+g) \geq \min(\mathrm{ord}_{D_i}(f), \mathrm{ord}_{D_i}(g)) \geq -a_i.$$

Since $\mathrm{div}(-f) = \mathrm{div}(f)$, we see that $\Gamma(U, \mathcal{O}_X(D))$ is a subgroup of $\mathbb{C}(X)$.

We next show that this is a module over $\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X)$. Given $f \in \Gamma(X, \mathcal{O}_X(D))$ and $g \in \Gamma(X, \mathcal{O}_X)$, we know that $\mathrm{div}(f) + D \geq 0$ and $\mathrm{div}(g) \geq 0$. Then

$$\mathrm{div}(gf) + D = \mathrm{div}(g) + \mathrm{div}(f) + D \geq 0$$

since a sum of effective divisors is effective. This proves that $gf \in \Gamma(X, \mathcal{O}_X(D))$ and gives the desired module structure.

Finally, we omit the proof that $\mathcal{O}_X(D)$ is a sheaf in the Zariski topology. □

Exercise 9.1. The trivial Weil divisor is denoted 0. Prove that $\mathcal{O}_X(0)$ coincides with the structure sheaf $\mathcal{O}_X$. Hint: Use Proposition 7.2.

We next show that linearly equivalent divisors give isomorphic sheaves.

Proposition 9.2. *If $D \sim E$ are linearly equivalent Weil divisors, then $\mathcal{O}_X(D)$ and $\mathcal{O}_X(E)$ are isomorphic as sheaves of $\mathcal{O}_X$-modules.*

Proof. By assumption, we have $D = E + \mathrm{div}(g)$ for some $g \in \mathbb{C}(X)^*$. Then

$$\begin{aligned} f \in \Gamma(X, \mathcal{O}_X(D)) &\iff \mathrm{div}(f) + D \geq 0 \\ &\iff \mathrm{div}(f) + E + \mathrm{div}(g) \geq 0 \\ &\iff \mathrm{div}(fg) + E \geq 0 \\ &\iff fg \in \Gamma(X, \mathcal{O}_X(E)). \end{aligned}$$

Thus multiplication by g induces an isomorphism $\Gamma(X, \mathcal{O}_X(D)) \simeq \Gamma(X, \mathcal{O}_X(E))$ which is clearly an isomorphism of $\mathcal{O}_X(X)$-modules.

The same argument works over any Zariski open set U, and the isomorphisms are easily seen to be compatible with the restriction maps. □

Weil Divisors on an Affine Variety. Now suppose that $X = \mathrm{Spec}(R)$ is affine and let K be the field of fractions of R. If D is a Weil divisor on $X = \mathrm{Spec}(R)$, then $\Gamma(X, \mathcal{O}_X(D))$ is an R-submodule of K. We first prove that this R-module is finitely generated.

Proposition 9.3. *Let D be a Weil divisor on the normal affine variety $X = \mathrm{Spec}(R)$. Then $\Gamma(X, \mathcal{O}_X(D))$ is a finitely generated R-module.*

Proof. We will prove the existence of $h \in R - \{0\}$ such that $h\Gamma(X, \mathcal{O}_X(D)) \subset R$. This will imply that $h\Gamma(X, \mathcal{O}_X(D))$ is an ideal of R and hence has a finite basis since R is Noetherian. It will follow immediately that $\Gamma(X, \mathcal{O}_X(D))$ is finitely generated as an R-module.

Write $D = \sum_{i=1}^{s} a_i D_i$. Since $\bigcup_{i=1}^{s} D_i$ is a proper subvariety of X, we can find $g \in R - \{0\}$ which vanishes on each D_i. Then $\mathrm{ord}_{D_i}(g) > 0$ for every i, so that $m\,\mathrm{ord}_{D_i}(g) > a_i$ for all i, provided $m \in \mathbb{Z}$ is sufficiently large. Since $\mathrm{div}(g) \geq 0$, it follows that $m\,\mathrm{div}(g) - D \geq 0$.

Now let $f \in \Gamma(X, \mathcal{O}_X(D))$. This means $\operatorname{div}(f) + D \geq 0$, and thus

$$\operatorname{div}(g^m f) = m \operatorname{div}(g) + \operatorname{div}(f) = m \operatorname{div}(g) - D + \operatorname{div}(f) + D \geq 0$$

since a sum of effective divisors is effective. By Proposition 7.2, we conclude that $g^m f \in \mathcal{O}_X(X) = R$. Hence $h = g^m \in R$ has the desired property. $\square$

A finitely generated R-submodule of K is called a *fractional ideal*. Thus Proposition 9.3 shows that $\Gamma(X, \mathcal{O}_X(D))$ is a fractional ideal.

Exercise 9.2. Let $D \geq 0$ be an effective divisor on the affine variety $X = \operatorname{Spec}(R)$. Prove that the fractional ideal $\Gamma(X, \mathcal{O}_X(-D))$ is an ordinary ideal (i.e., $\Gamma(X, \mathcal{O}_X(-D)) \subset R$). Hint: As usual, you will use Proposition 7.2.

We next show that the R-module $\Gamma(X, \mathcal{O}_X(D))$ determines the entire sheaf $\mathcal{O}_X(D)$. Recall that if $g \in R$ is nonzero, the Zariski open set X_g defined by the nonvanishing of g is $\operatorname{Spec}(R_g)$, where $R_g = \{a/g^m \mid a \in R,\ m \geq 0\}$ is the localization of R at g.

Proposition 9.4. *Let D be a Weil divisor on the normal affine variety $X = \operatorname{Spec}(R)$. If $g \in R$ is nonzero, then*

$$\Gamma(X_g, \mathcal{O}_X(D)) = \Big\{ \frac{f}{g^m} \mid f \in \Gamma(X, \mathcal{O}_X(D)),\ m \geq 0 \Big\}. \tag{9.1}$$

Proof. Let $D = \sum_{i=1}^s a_i D_i$ and write $\{1, \ldots, s\}$ as a disjoint union $I \cup J$ where $D_i \cap X_g \neq \emptyset$ for $i \in I$ and $D_j \subset \mathbf{V}(g)$ for $j \in J$.

Suppose that $h \in \Gamma(X_g, \mathcal{O}_X(D))$, so that $(\operatorname{div}(h) + D)\big|_{X_g} \geq 0$. Thus $\operatorname{ord}_{D_i}(h) \geq -a_i$ for $i \in I$. Notice that there is no constraint on $\operatorname{ord}_{D_j}(h)$ for $j \in J$. However, we do know that g vanishes on D_j for $j \in I$, so that $\operatorname{ord}_{D_j}(g) > 0$. Then we can pick $m \in \mathbb{Z}$ sufficiently large so that

$$m \operatorname{ord}_{D_j}(g) + \operatorname{ord}_{D_j}(h) > 0 \quad \text{for } j \in J.$$

Since $\operatorname{div}(g) \geq 0$, it follows easily that $\operatorname{div}(g^m h) + D \geq 0$ on X. Thus $f = g^m h \in \Gamma(X, \mathcal{O}_X(D))$, and then $h = f/g^m$ has the desired form. From here, the proposition follows easily. $\square$

Since the open sets X_g for $g \in R - \{0\}$ form a basis for the Zariski topology of $X = \operatorname{Spec}(R)$, Proposition 9.4 shows that the sheaf $\mathcal{O}_X(D))$ is uniquely determined by its global sections.

Coherent Sheaves. The right-hand side of (9.1) is the *localization* of $\Gamma(X_g, \mathcal{O}_X(D))$ at g. More generally, given any finitely generated R-module M, one can define its localization M_g, and then one gets a unique sheaf $\widetilde{M}$ on $X = \operatorname{Spec}(R)$ such that

$$\Gamma(X_g, \widetilde{M}) = M_g$$

for any $g \in R - \{0\}$. See [5, II.5] for details.

For example, if $X = \operatorname{Spec}(R)$, Theorem 3.1 of §3 implies $\widetilde{R} = \mathcal{O}_X$, and if D is a Weil divisor on X, Proposition 9.4 implies $\widetilde{M} = \mathcal{O}_X(D)$ for $M = \Gamma(X, \mathcal{O}_X(D))$.

This leads to the following general definition. Suppose that $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on an arbitrary variety X. Then $\mathcal{F}$ is *coherent* if there is an affine open cover $\{U_i\}_{i \in I}$ of X such that for every $i \in I$, there is a finitely generated $\mathcal{O}_X(U_i)$-module M_i such that

$$\mathcal{F}\big|_{U_i} = \widetilde{M_i}.$$

The simplest example of a coherent sheaf is $\mathcal{O}_X$. Furthermore, the above discussion shows that if D is a Weil divisor on a normal variety X, then $\mathcal{O}_X(D)$ is also coherent. We will learn more properties of $\mathcal{O}_X(D)$ in the next section.

§10. Invertible Sheaves and Line Bundles

We next discuss an especially nice class of sheaves.

Invertible Sheaves. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$ modules on a variety X. Then $\mathcal{F}$ is *invertible* if it is locally trivial, i.e., if there is a Zariski open cover of $\{U_i\}_{i\in I}$ of X such that $\mathcal{F}\big|_{U_i} \simeq \mathcal{O}_X\big|_{U_i}$.

It follows immediately that $\mathcal{O}_X$ is invertible. A more interesting result is the following characterization of when the sheaves $\mathcal{O}_X(D)$ from §4 are invertible.

Theorem 10.1. *Let D be a Weil divisor on a normal variety X. Then $\mathcal{O}_X(D)$ is an invertible sheaf if and only if D is a Cartier divisor.*

Proof. First suppose that D is Cartier. Since invertibility is a local property and D is locally principal, we may assume that $X = \mathrm{Spec}(R)$ is affine and $D = \mathrm{div}(f)$ for $f \in K$. Then $D \sim 0$, so that by Proposition 9.2, we have

$$\mathcal{O}_X(D) \simeq \mathcal{O}_X(0) = \mathcal{O}_X,$$

where the last equality is by Exercise 9.1.

Going the other way, suppose that $\mathcal{O}_X(D)$ is invertible. We need to prove that D is locally principal. By restricting to a suitable affine open subset, we can assume that $X = \mathrm{Spec}(R)$ and that $\mathcal{O}_X \simeq \mathcal{O}_X(D)$. Taking global sections, we get an isomorphism

$$R \simeq \Gamma(X, \mathcal{O}_X(D)) \subset K.$$

Under this isomorphism, suppose that $1 \in R$ maps to $1/g \in \Gamma(X, \mathcal{O}_X(D))$. The proof of Proposition 9.2 shows that if we set $E = D - \mathrm{div}(g)$, then $g\Gamma(X, \mathcal{O}_X(D)) = \Gamma(X, \mathcal{O}_X(E))$. Thus $\Gamma(X, \mathcal{O}_X(E)) = R$, so that $\mathcal{O}_X = \mathcal{O}_X(E)$. If we can show that this forces $E = 0$, then $D = \mathrm{div}(g)$ will follow, proving that D is locally principal and hence Cartier.

Thus we may assume $\mathcal{O}_X = \mathcal{O}_X(E)$. Then $1 \in \Gamma(X, \mathcal{O}_X(E))$, which implies $E \geq 0$. If $E \neq 0$, then some irreducible hyperface Y appears in E with positive coefficient. Observe that any affine open subset of X which meets Y has the same property. By Exercise 10.1 below, we can then assume that $\mathrm{div}(h) = Y$ for some $h \in R$. It follows that $\mathrm{div}(1/h) + E \geq 0$, so that $1/h \in \Gamma(X, \mathcal{O}_X(E)) = \Gamma(X, \mathcal{O}_X) = R$. Since h is also in R, this implies that h is invertible, which means $\mathrm{div}(h) = 0$. This contradicts $\mathrm{div}(h) = Y$ and proves $E = 0$, as desired. □

Exercise 10.1. Let Y be an irreducible hypersurface in a normal variety X. The goal of this exercise is to find an affine open subset U and a rational function $h \in \mathbb{C}(X)$ such that $Y \cap U \neq \emptyset$ and $\mathrm{div}(h)\big|_U = Y \cap U$.

a. Explain why there is $h \in \mathcal{O}_{X,Y}$ with $\mathrm{ord}_Y(h) = 1$.
b. Show that h from part a has the following two properties:
   - $\mathrm{div}(h) = Y + \sum_{j=1}^r b_j E_j$, where the E_j are distinct from Y.
   - h is defined on a Zariski open set U' such that $U' \cap Y \neq \emptyset$.
c. Show that $U' - (E_1 \cup \cdots \cup E_r)$ is nonempty and has nonempty intersection with Y.
d. Now show that the desired affine open subset U exists.

We can also improve Proposition 492 as follows.

Exercise 10.2. Let D and E be Cartier divisors on a normal variety X. Then $D \sim E$ if and only if $\mathcal{O}_X(D) \simeq \mathcal{O}_X(E)$ as $\mathcal{O}_X$-modules. Hint: Adapt the argument of Theorem 10.1.

A deeper result is the following. See Proposition 6.15 of [5, II.6] for a proof.

Theorem 10.2. *Let X be a normal variety. Then every invertible sheaf on X is isomorphic to $\mathcal{O}_X(D)$ for some Cartier divisor D on X.*

We remark that invertible sheaves are sometimes called *locally free sheaves of rank one*.

The Picard Group. Given invertible sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, one easily proves that

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \quad \text{and} \quad \mathcal{F}^\vee = \underline{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \tag{10.1}$$

are also invertible. It is also easy to show that the canonical map $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee \to \mathcal{O}_X$ induces an isomorphism

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee \simeq \mathcal{O}_X,$$

which explains the name "invertible". These properties show that the set of isomorphism classes of invertible sheaves on X has a natural group structure under tensor product. We call

$$\mathrm{Pic}(X) = \{\text{isomorphism classes of invertible sheaves on } X\}. \tag{10.2}$$

the *Picard group* of X,

Since we already defined $\mathrm{Pic}(X)$ in (8.1) of §8, we need to explain why these definitions are equivalent. We begin with the following important result, whose proof we omit (see Proposition 6.13 of [5, II.6] for a proof).

Theorem 10.3. *If D and E are Cartier divisors on a normal variety X, then there are canonical isomorphisms*

$$\mathcal{O}_X(D+E) \simeq \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)$$
$$\mathcal{O}_X(-D) \simeq \mathcal{O}_X(D)^\vee.$$

If we combine Theorem 10.2 and 10.3, we get a surjective homomorphism $\mathrm{Div}(X) \to \mathrm{Pic}(X)$, and Exercise 5.2 shows that the kernel is $\mathrm{Div}_0(X)$. We conclude that for normal varieties, the two definitions coincide. However, the definition given in (10.2) is more general, since it makes sense for any variety X.

We should also note that one can define $\mathrm{Pic}(X)$ using sheaf cohomology. Here is the basic idea. Let X be a normal variety (for simplicity), and consider the exact sequence of sheaves

$$1 \to \mathcal{O}_X^* \to \mathbb{C}(X)^* \to \mathbb{C}(X)^*/\mathcal{O}_X^* \to 1. \tag{10.3}$$

In §8, we mentioned that $\mathrm{Div}(X) = H^0(X, \mathbb{C}(X)^*/\mathcal{O}_X^*)$. Taking sheaf cohomology, (10.3) gives the *long exact sequence*

$$0 \to H^0(X, \mathcal{O}_X^*) \to H^0(X, \mathbb{C}(X)^*) \to H^0(X, \mathbb{C}(X)^*/\mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*) \to H^1(X, \mathbb{C}(X)^*) \to$$

One can show that $H^1(X, \mathbb{C}(X)^*) = 0$, and then the above long exact sequence reduces to

$$1 \to \mathcal{O}_X(X)^* \to C(X)^* \to \mathrm{Div}(X) \to H^1(X, \mathcal{O}_X^*) \to 0.$$

Comparing this to Exercise 8.4, we conclude that $H^1(X, \mathcal{O}_X^*) = \operatorname{Pic}(X)$. This is the sheaf-theoretic definition of the Picard group.

Rank One Reflexive Sheaves. Now suppose that D is a Weil divisor on a normal variety X. If D is not Cartier, then we know that $\mathcal{O}_X(D)$ is not invertible. So what kind of sheaf is it?

Given any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we can define $\mathcal{F}^\vee$ as in (10.1), and there is a canonical map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$. Then we say that $\mathcal{F}$ is *reflexive of rank one* if:

- There is a nonempty Zariski open set U such that $\mathcal{F}\big|_U$ is trivial.
- $\mathcal{F}$ is torsion-free.
- The map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism.

Any invertible sheaf is reflexive. Of more interest is the following result. A proof can be found in [2, Chapter VII] and [8, Appendix to §1].

Proposition 10.4. *If D is a Weil divisor on a normal variety X, then $\mathcal{O}_X(D)$ is a reflexive sheaf of rank one.*

The dual of a reflexive sheaf of rank one is again reflexive of rank one, though the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ of reflexive sheaves of rank one need not be reflexive of rank one. However, the double dual

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{\vee\vee} \tag{10.4}$$

is reflexive of rank one. Furthermore, if D and E are Weil divisors on X, then

$$\mathcal{O}_X(D+E) \simeq (\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E))^{\vee\vee}.$$

One can also show that up to isomorphism, every reflexive sheaf of rank one on X comes from a Weil divisor on X. It follows that the class group $\operatorname{Cl}(X)$ can be regarded as the group of isomorphism classes of rank one reflexive sheaves under the product (10.4). Details of all of this can be found int [2, Chapter VII] and [8, Appendix to §1].

In most of algebraic geometry, invertible sheaves are more important than rank one reflexive sheaves. However, there are situations where rank one reflexive sheaves occur naturally. An example is given by the *canonical sheaf* of a Cohen-Macaulay variety X, which is only reflexive of rank one (unless the variety is Gorenstein, in which case the canonical sheaf is invertible). The canonical sheaf plays an important role in duality theory.

Line Bundles. A *line bundle* over a variety X consists of a map of varieties $\pi : L \to X$ such that X has an open cover $\{U_i\}_{i\in I}$ with the following two properties:

- For each $i \in I$, here is an isomorphism $f_i : \pi^{-1}(U_i) \simeq U_i \times \mathbb{C}$ such that $\pi = \pi_1 \circ f_i$, where $\pi_1 : U_i \times \mathbb{C} \to U_i$ is projection on the first factor.
- For each pair $i, j \in I$, then there is $g_{ij} \in \mathcal{O}_X(U_i \cap U_j)^*$ such that the composition
$$f_j \circ f_i^{-1} : (U_i \cap U_j) \times \mathbb{C} \to (U_i \cap U_j) \times \mathbb{C}$$
is given by $(x, \lambda) \mapsto (x, g_{ij}(x)\,\lambda)$.

Since the $g_{ij} \in \mathcal{O}_X(U_i \cap U_j)^*$ are built from $f_j \circ f_i^{-1}$, it follows easily that they satisfy the *cocycle condition*

$$g_{ik}(x) = g_{ij}(x) g_{jk}(x) \quad \text{for} \quad i, j, k \in I\; x \in U_i \cap U_j \cap U_k. \tag{10.5}$$

The *trivial line bundle* is given by $\pi_1 : X \times \mathbb{C} \to X$, where π_1 is projection on the first factor.

Given $x \in X$ and a line bundle $\pi : L \to X$, we call $L_x = \pi^{-1}(x)$ the *fiber* of L over x. If $x \in U_i$, we can use f_i to define an isomorphism $L_x \simeq \mathbb{C}$. If we also have $x \in U_j$, then we get a different isomorphism $L_x \simeq \mathbb{C}$, but the two are related by multiplication by $g_{ij}(x)$. It follows that L_x has a natural structure as a 1-dimensional vector space, i.e., a complex line. Since L is the union of the L_x, this explains the term "line bundle".

The Sheaf of Sections of a Line Bundle. Let $\pi : L \to X$ be a line bundle over X. If $U \subset X$ is Zariski open, then a *section* of L over U is a morphism $s : U \to L$ such that $\pi \circ s(x) = x$ for all $x \in U$. Then set

$$\Gamma(U, L) = H^0(U, L) = \{\text{all sections of } L \text{ over } U\}. \tag{10.6}$$

Since the fibers are vector spaces, we can add sections and mutiply them by elements of $\mathcal{O}_X(U)$. It follows that (10.6) defines a sheaf of $\mathcal{O}_X$-modules. We will denote this sheaf by $\mathcal{O}_X(L)$.

Exercise 10.3. Let $\pi_1 : X \times \mathbb{C} \to X$ be the trivial bundle.

a. Show that a section over $U \subset X$ is described by $s(x) = (x, f(x))$, $x \in U$, for a unique $f \in \mathcal{O}_X(U)$.
b. Show that the sheaf defined by (10.6) is $\mathcal{O}_X$.

Now let L be any line bundle over X. Since locally L looks like $U_i \times \mathbb{C}$, Exercise 10.3 shows that locally, the sheaf $\mathcal{O}_X(L)$ looks like $\mathcal{O}_{U_i} \simeq \mathcal{O}_X\big|_{U_i}$. We have thus proved the following result.

Proposition 10.5. *If L is a line bundle over X, then $\mathcal{O}_X(L)$ is an invertible sheaf.*

We can also reverse this process by showing that *every* invertible sheaf is the sheaf of sections of some line bundle. In the special case when X is normal, we can do this as follows. Suppose that $\mathcal{L}$ is an invertible sheaf. By Theorem 10.2, $\mathcal{L} \simeq \mathcal{O}_X(D)$ for some Cartier divisor D. Then let $\{(U_i, f_i)\}_{i\in I}$ be local data for D, so that $\operatorname{div}(f_i)\big|_{U_i} = D\big|_{U_i}$ for all i.

With this set-up, let $g_{ij} = f_i/f_j$, and note that $g_{ij} \in \mathcal{O}_X(U_i \cap U_j)^*$ by Exercise 8.3. Furthermore, it is obvious that the g_{ij} satisfy cocycle condition (10.5). We saw in §3 how we can construct X from the U_i by gluing U_i and U_j together along $U_i \cap U_j$. In the same way, we can glue $U_i \times \mathbb{C}$ and $U_j \times \mathbb{C}$ together by identifying

$$(x, \lambda) \longleftrightarrow (x, g_{ij}(x)\,\lambda), \tag{10.7}$$

where

$$\begin{aligned}(x, \lambda) &\in (U_i \cap U_j) \times \mathbb{C} \subset U_i \times \mathbb{C}\\ (x, g_{ij}(x)\,\lambda) &\in (U_i \cap U_j) \times \mathbb{C} \subset U_j \times \mathbb{C}.\end{aligned}$$

The cocycle condition (10.5) shows that thes identifications satisfy the compatibility conditions from the subsection "Gluing Together Affine Varieties" in §3. It follows that we can glue together the $U_i \times \mathbb{C}$ to get a variety L. In the same way, the projections $U_i \times \mathbb{C} \to U_i$ patch together to give a morphism $\pi : L \to X$. We will omit the proof of the following proposition.

Proposition 10.6. *$\pi : L \to X$ is a line bundle whose sheaf of sections is isomorphic to the invertible sheaf $\mathcal{L} \simeq \mathcal{O}_X(D)$ we began with.*

It follows that we have three closely related objects: Cartier divisors, invertible sheaves, and line bundles. In algebraic geometry, it is customary (though slightly inaccurate) to use the terms "invertible sheaf" and "line bundle" interchangeably.

The Zero Divisor of a Section. Suppose that the sheaf of sections of a line bundle L is the invertible sheaf $\mathcal{O}_X(D)$, where D is a Cartier divisor. Then we can think of a global section in two very different ways: as a section $s : X \to L$ such that $\pi \circ s = 1_X$, and as a rational function $f \in \mathbb{C}(X)^*$ such that $\mathrm{div}(f) + D \geq 0$. How are these related?

The easiest way to see the link between these notions of "global section" is to define the *zero divisor* of a nonzero section $s : X \to L$. Given such an s, consider an open covering $\{U_i\}_{i\in I}$ which trivializes the bundle. Then, using the restriction of s to U_i, we get the composition

$$s_i : U_i \to \pi^{-1}(U_i) \simeq U_i \times \mathbb{C} \to \mathbb{C}. \tag{10.8}$$

This is a morphism, so that $s_i \in \mathcal{O}_X(U_i)$. Furthermore, one checks that $s_i = g_{ij}s_j$ on $U_i \cap U_j$. Since $g_{ij} \in \mathcal{O}_X(U_i \cap U_j)^*$, It follows easily that the divisors $\mathrm{div}(s_i)$ on U_i patch together to give a divisor on X. This divisor is clearly locally principal (it equals $\mathrm{div}(s_i)$ on U_i). Thus we get a Cartier divisor

$$\mathrm{div}_0(s) \in \mathrm{Div}(X).$$

Furthermore, $\mathrm{div}_0(s) \geq 0$ since each $s_i \in \mathcal{O}_X(U_i)$. This relates to the global sections of $\mathcal{O}_X(D)$ as follows.

Theorem 10.7. *Let L be the line bundle corresponding to the invertible sheaf $\mathcal{O}_X(D)$, and suppose that $s \in \Gamma(X, L) - \{0\}$ corresponds to $f \in \Gamma(X, \mathcal{O}_X(D)) - \{0\}$. Then*

$$\mathrm{div}_0(s) = \mathrm{div}(f) + D.$$

Proof. First note that both sides of the equation are effective divisors. Given $f \in \mathbb{C}(X)^*$ with $\mathrm{div}(f) + D \geq 0$, we can define a section s of L as follows. We constructed L using the local data $\{(U_i, f_i)\}_{i\in I}$ for D. Then $D\big|_{U_i} = \mathrm{div}(f_i)\big|_{U_i}$, so that

$$\mathrm{div}(ff_i)\big|_{U_i} = (\mathrm{div}(f) + D)\big|_{U_i} \geq 0.$$

If we set $s_i = ff_i$, then Proposition 7.2 shows that $s_i \in \mathcal{O}_X(U_i)$. Furthermore, the construction of L shows that the sections $U_i \to U_i \times \mathbb{C}$ defined by $x \mapsto (x, s_i(x))$ patch to give a section s of L over X. (This is part of the proof of Proposition 10.6.) Since $\mathrm{div}_0(s)$ is constructed by patching together the divisors $\mathrm{div}(s_i) = \mathrm{div}(ff_i)$, it follows easily that $\mathrm{div}_0(s) = \mathrm{div}(f) + D$, as claimed. □

The divisor $\mathrm{div}_0(s)$ tells us where the section s vanishes. However, being a divisor, $\mathrm{div}_0(s)$ records more than just the hypersurfaces $Y \subset X$ where s is zero—the coefficient of Y in $\mathrm{div}_0(s)$ also tells us to what order s vanishes on Y.

Exercise 10.4. Let L be a line bundle over X.

a. Show that the divisors $\mathrm{div}_0(s)$ for $s \in \Gamma(X, L) - \{0\}$ are all linearly equivalent.
b. Let D be an effective Cartier divisor on X which is linearly equivalent to $\mathrm{div}_0(s)$ for some $s \in \Gamma(X, L) - \{0\}$. Prove that $D = \mathrm{div}_0(t)$ for some $t \in \Gamma(X, L)$

Given a line bundle L over X, the set of effective divisors

$$|L| = \{\mathrm{div}_0(s) \mid s \in \Gamma(X, L) - \{0\}\}$$

is called a *complete linear system*. This terminology is justified by part b of the Exercise 10.4.

Exercise 10.5. If L is a line bundle on a complete variety X. Prove that $|L|$ can be identified with the projective space $\mathbb{P}(\Gamma(X, L))$. Hint: Exercise 3.4 will be useful.

Finally, we discuss the "quotient" of two sections of a line bundle. If s, t are nonzero sections of L, then for each $x \in X$, $s(x)$ and $t(x)$ are elements of the one-dimensional vector space L_x. This space doesn't have a canonical basis, so we can't regard $s(x)$ and $t(x)$ as numbers. But if $t(x) \neq 0$, then the "quotient" $s(x)/t(x)$ makes sense: it the unique number λ such that $s(x) = \lambda t(x)$. This suggests that s/t should be a rational function on X.

Exercise 10.6. Let s, t be nonzero sections of L over X, and let $\{U_i\}_{i \in I}$ be an open cover of X which trivializes L.

a. By working on U_i, show that $s/t = s_i/t_i$, where s_i is as in (10.8) and t_i is defined similarly.
b. Explain why $s_i/t_i = s_j/t_j$ as rational functions on $U_i \cap U_j$.

Part b gives a well-defined element of $\mathbb{C}(X)^*$ which we denote s/t.

Exercise 10.7. Suppose that L is the line bundle built from the Cartier divisor D on X. Let s, t be nonzero sections of L over X which correspond to $f, g \in \Gamma(X, \mathcal{O}_X(D))$. Prove that the rational function s/t of Exercise 10.6 is given by f/g.

Invertible Sheaves on Projective Space. Let $x_0, \ldots, x_n$ be homogeneous coordinates on $\mathbb{P}^n$. Recall from §2 that $\mathbb{P}^n$ is covered by the open sets $U_i = \mathbb{P}^n - \mathbf{V}(x_i)$ and that

$$\mathbb{C}(\mathbb{P}^n) = \Big\{ \frac{f}{g} \mid f, g \in \mathbb{C}[x_0, \ldots, x_n] \text{ homogeneous of equal degree, } g \neq 0 \Big\}.$$

Now let $H = \mathbf{V}(x_0) \subset \mathbb{P}^n$. This is clearly a divisor, and is Cartier since $\mathbb{P}^n$ is smooth. Our goal is to determine the global sections of $\mathcal{O}_{\mathbb{P}^n}(dH)$ for $d > 0$.

Exercise 10.8. Show that $\{(U_i, x_0^d/x_i^d)\}_{0 \leq i \leq n}$ is local data for dH.

We can now describe the global sections of $\mathcal{O}_{\mathbb{P}^n}(dH)$.

Proposition 10.8. *If $d > 0$, then the global sections of $\mathcal{O}_{\mathbb{P}^n}(dH)$ are*

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(dH)) = \Big\{ \frac{f}{x_0^d} \mid f \text{ is homogeneous of degree } d \Big\}.$$

Proof. Let $f/g \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(dH))$, where f and g are relatively prime. Then $\operatorname{div}(f/g) + dH \geq 0$. If we restrict to U_i, then this becomes $(\operatorname{div}(f/g) + \operatorname{div}(x_0^d/x_i^d))\big|_{U_i} \geq 0$. Equivalently,

$$\operatorname{div}(f/g \cdot x_0^d/x_i^d)\big|_{U_i} \geq 0,$$

so that $f/g \cdot x_0^d/x_i^d \in \mathcal{O}_{\mathbb{P}^n}(U_i)$. Can can think of U_i as a copy of $\mathbb{C}^n$ with variables $\frac{x_j}{x_i}$ for $j \neq i$. If f, g have degree m, then we can write $f/g \cdot x_0^d/x_i^d$ as

$$\frac{f\big(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\big)}{g\big(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\big)} \cdot \Big(\frac{x_0}{x_i}\Big)^d. \tag{10.9}$$

For each i, this must be a polynomial in $\frac{x_j}{x_i}$ for $j \neq i$. When $i = 0$, the second factor in (5.9) is 1, which means that the denominator of the first factor must be constant since f and g are relatively prime. Multiplying f and g by suitable constants, we can assume $g\big(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\big) = 1$, and then $g = x_0^m$ follows since g is homogeneous of degree m.

Furthermore, if we then consider (10.9) with $i \neq 0$, then $f\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)\left(\frac{x_0}{x_i}\right)^{d-m}$ is a polynomial in $\frac{x_j}{x_i}$ for $j \neq i$. Since f and $g = x_0^m$ are relatively prime, it follows that $d \geq m$. Then multiplying f and g by x_0^{d-m} shows that f and g have the desired form.

Conversely, we need to show that f/x_0^d lies in $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(dH))$ whenever f is homogeneous of degree d. This follows easily using the above methods. We omit the details. $\square$

Exercise 10.9. Show that $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(dH)) = \{0\}$ if $d < 0$.

It is customary to write $\mathcal{O}_{\mathbb{P}^n}(dH)$ as $\mathcal{O}_{\mathbb{P}^n}(d)$. Then, up to isomorphism, the global sections of $\mathcal{O}_{\mathbb{P}^n}(d)$ form the vector space of homogeneous polynomials of degree d.

Exercise 10.10. Show that every invertible sheaf on $\mathbb{P}^n$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(d)$ for some $d \in \mathbb{Z}$.

References

1. M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, MA, 1969.
2. N. Bourbaki, *Algèbre commutative*, Hermann, Paris, 1965. English translation, Addison-Wesley, Reading, MA, 1972.
3. D. Cox, J. Little and D. O'Shea, *Ideals, Varieties, and Algorithms*, Second Edition, Springer-Verlag, New York-Berlin-Heidelberg, 1997.
4. D. Eisenbud and J. Harris, *The Geometry of Schemes*, Springer-Verlag, New York-Berlin-Heidelberg, 2000.
5. R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York-Berlin-Heidelberg, 1977.
6. H. Matsumura, *Commutative Ring Theory*, Cambridge Univ. Press, Cambridge, 1986.
7. D. Prill, *Local classification of quotients of complex manifolds by discontinuous groups*, Duke Math. J. **34** (1967), 375–386.
8. M. Reid, *Canonical 3-folds*, in *Journées de géométrie algébrique d'Angers* (A. Beauville, ed.), Sijthoff &Noordhoff, Alphen aan den Rijn, 1980, 273–310.
9. J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier **6** (1956), 1–42.
10. B. Sturmfels, *Algorithms in Invariant Theory*, Springer-Verlag, New York-Berlin-Heidelberg, 1993.
11. O. Zariski and P. Samuel, *Commutative Algebra*, Volume I, Van Nostrand, Princeton, NJ, 1958. Reprint by Springer-Verlag, New York-Berlin-Heidelberg, 1975.