Surfaces

I. Curves & Moving Lines
II. Surfaces and Moving Quadrics

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Rectangular Case

Consider $a, b, c, d \in R = k[s, t]$, built from monomials $s^i t^j, 0 \leq i \leq n, 0 \leq j \leq m$.

Make bihomogeneous of degree (n, m) and assume no basepoints. Get

$$\phi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

Let $S =$ image of ϕ. If ϕ has degree 1, then

$$\deg(\phi)\deg(S) = n^2 - \sum e_p \quad \text{for } \mathbb{P}^2$$

gives $\deg(S) = 2nm$ by Bezout for $\mathbb{P}^1 \times \mathbb{P}^1$.

The Theorem

Theorem (C-Goldman-Zhang): Assume a, b, c, d have degree (n, m), no basepoints, degree 1, and no moving planes of degree $(n-1, m-1)$. Then there are nm moving quadrics of degree $(n-1, m-1)$ such that S is defined by

$$\det\begin{pmatrix}\text{matrix of moving quadrics wrt} \\ \text{monomials deg } (n-1, m-1)\end{pmatrix}.$$

The matrix M is $nm \times nm$ and its entries are quadratic in x, y, z, w. So $\det(M)$ has deg $2nm$.

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- Equations of Rees algebra include

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  Ignore mov plane; mov quad give implicit.
- Theorem covers generic case:
  No basepoints, deg 1, and

$$R^4_{n-1,m-1} \xrightarrow{(a,b,c,d)} R_{2n-1,2m-1}$$

  Each has dim $4nm$; generically an iso.

Comments

- Hong-Simis-Vasconcelos II (Equations of Almost Complete Intersections) considers $\phi : \mathbb{P}^{d-1} \to \mathbb{P}^d$ defined by $(a_1, \ldots, a_d, a)$, where $a_1, \ldots, a_d$ form a regular sequence.
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  - Our case is very different!

Our Case

$\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ defined by a, b, c, d.

- a, b, c can never be a regular sequence:

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- Replacement: Assume $a = b = c = 0$ have no solutions in $\mathbb{P}^1 \times \mathbb{P}^1$. The Koszul complex

$$0 \to R(-3n, -3m) \to R(-2n, -2m)^3 \to R(-n, -m)^3 \to I \to 0$$

is exact up to B torsion, $B = \langle st, sv, ut, uv \rangle$ the irrelevant ideal. (Used in multigraded regularity).

Sheafify

Set $\mathcal{O} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$ and sheafify the Koszul complex:

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This is exact when $a = b = c = 0$ have no solutions in $\mathbb{P}^1 \times \mathbb{P}^1$. We will:

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- Take cohomology
- Use vanishing theorems

Moving Quadrics

Moving quadrics $= \mathrm{Syz}(I^2)$. Write one as

$$F = Ax^2 + Bxy + \cdots + Jw^2.$$

where $A, \ldots, J \in R = k[s,u;t,v]$, so

$$Aa^2 + Bab + \cdots + Jd^2 = 0.$$

Note $\mathrm{Syz}(a^2, \ldots, d^2)$ is the kernel of

$$\underbrace{R^{10}_{n-1,m-1}}_{10nm} \xrightarrow{(a^2,\ldots,d^2)} \underbrace{R_{3n-1,3m-1}}_{9nm}$$

Thus nm mov quad $\iff$ this map is onto.

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- There is a lot going on here.

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  - The surface S has degree $2nm$.
  - $\det(M)$ vanishes on S.
- Show $\det(M)$ is not identically zero.

Proof of Onto

Assume a, b, c don't vanish simultaneously. Take the $9nm \times 9nm$ minor of Φ given by $a^2, \ldots, cd$ and assume its kernel is nontrivial. Then:

$$c_1a^2 + \cdots + c_9cd = 0$$

for $c_1, \ldots, c_9 \in R_{n-1,m-1}$. Rewrite:

$$(c_1a+\cdots+c_4d)a+(c_5b+\cdots+c_7d)b+(c_8c+c_9d)c = 0$$

Suppose
$$\begin{aligned} c_1a + \cdots + c_4d &= h_1c + h_2b \\ c_5b + \cdots + c_7d &= -h_2a + h_3c \\ c_8c + c_9d &= -h_1a - h_3b \end{aligned}$$

Last line gives mov line deg $(n-1, m-1)$!

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- Twist exact sheaf seq by $\mathcal{O}(3n-1, 3m-1)$:

$$0 \to \mathcal{O}(-1,-1) \to \mathcal{O}(n-1, m-1)^3 \to \mathcal{O}(2n-1, 2m-1)^3 \to \mathcal{O}(3n-1, 2m-1) \to 0$$

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- Since $\mathcal{O}(-1,-1)$ has no cohomology,

$$R^3_{n-1,m-1} \longrightarrow R^3_{2n-1,2m-1} \longrightarrow R_{3n-1,3m-1}$$

is exact in the middle. Onto is proved!

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- Then

$$\det(M) = \begin{pmatrix} w^2 & * & \cdots & * \\ * & w^2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & w^2 \end{pmatrix} = w^{2nm} + \cdots$$

QED

Final Comments

- The Koszul complex

$$0 \to R(-3n,-3m) \to R(-2n,-2m)^3 \to R(-n,-m)^3 \to R \to 0$$

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- Also recall that for $i > 0$,

$$H^{i+1}_B(R) = \bigoplus_{\alpha,\beta} H^i(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(\alpha, \beta)).$$

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- For a more sophisticated approach, see Botbol's lecture next Thursday.

Postscript

Craig Huneke made two useful comments:

- The local cohomology $H^i_B(R)$ is computed by Mayer-Vietoris since $\mathbf{V}(B)$ is union to two coord. planes meeting transverseley in $\mathbb{A}^4$.
- Since we assume $I_{2n-1,2m-1} = R_{2n-1,2m-1}$ and I^2 is gen. by $I_{n,m}I_{n,m}$, onto follows from

$$\begin{aligned} I^2_{3n-1,3m-1} &= I_{n,m}I_{2n-1,2m-1} = I_{n,m}R_{2n-1,2m-1} \\ &= I_{n,m}R_{n-1,m-1}R_{n,m} = I_{2n-1,2m-1}R_{n,m} \\ &= R_{2n-1,2m-1}E_{n,m} = R_{3n-1,3m-1} \end{aligned}$$

Need the special $9nm \times 9nm$ minor for $\det(M) \neq 0$.