What is the Multiplicity of a Base Point?

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Outline

- Bézout's Theorem
- Degree of a Rational Surface
- Serre's Definition
- Hilbert-Samuel Definition
- Combinatorial Computation
- Algebraic Computation
- Proof of Degree Formula
- Other Stuff (Time Permitting)

Bézout's Theorem

Suppose that $C, D \subset \mathbf{P}^2$ are curves of degrees n, m with no common components. Then

$$nm = \sum_{p \in C \cap D} I_p(C, D)$$

where

$$I_p(C, D) = \dim \mathcal{O}_{\mathbf{P}^2, p} / \langle \tilde{f}, \tilde{g} \rangle$$

and $\tilde{f}, \tilde{g}$ are local equations of C, D near p.

Degree of a Rational Surface

$\varphi : \mathbf{P}^2 \dashrightarrow \mathbf{P}^3$ is defined by a, b, c, d of degree n, no common factors.

- $Z = \{p \mid a(p) = \cdots = 0\}$ is the base point locus.
- $S = \overline{\varphi(\mathbf{P}^2 \setminus Z)}$ is the image.

We assume $\dim S = 2$.

Degree Formula:

$$n^2 = \deg S \cdot \deg \varphi + \sum_{p \in Z} m(p).$$

$m(p)$ is the "multiplicity" of p.

Naive guess:

$m(p) = \dim \mathcal{O}_{\mathbf{P}^2,p}/I_p = \dim R_p/I_p$,

where $I_p = \langle \tilde{a}, \ldots, \tilde{d} \rangle$

Counterexample:

$a = s^2u + t^3$, $b = t^2u + s^3$,
$c = stu$, $d = s^2u$ gives φ
with basepoint $p = (0, 0, 1)$
and $I_p = \langle s^2, st, t^2 \rangle$.

The naive guess implies

$$m(p) = \dim R_p/I_p = 3.$$

However:

A Gröbner basis calculuation shows that the image surface $S \subset \mathbf{P}^3$ has degree 5.

Consequence:

Since $n = 3$, the Degree Formula gives

$$3^2 = 5 \cdot 1 + m(p),$$

so that

$$m(p) = 4$$

in this case.

Serre's Definition

R local ring, m maximal ideal.
Assume R contains $k = R/m$.

Then R-modules M, N with

$$\dim_k M \otimes_R N < \infty$$

have **intersection multiplicity** $\chi(M, N)$ defined by

$$\sum_{i \geq 0} (-1)^i \dim_k \mathrm{Tor}_i^R(M, N).$$

Example: Bézout Situation

Let $R = \mathcal{O}_{\mathbf{P}^2,p}$, $M = R/\langle\tilde{f}\rangle$, $N = R/\langle\tilde{g}\rangle$. Then

$$0 \longrightarrow R \xrightarrow{\tilde{f}} R \longrightarrow M \longrightarrow 0$$

gives

$$\to \mathrm{Tor}_1^R(R, N) \to \mathrm{Tor}_1^R(M, N) \\ \to N \xrightarrow{\tilde{f}} N \to M \otimes_R N \to 0$$

However, no common component implies

$$N = R/\langle\tilde{g}\rangle \xrightarrow{\tilde{f}} N = R/\langle\tilde{g}\rangle$$

is one-to-one.

Thus $\mathrm{Tor}_i^R(M,N) = 0$ for $i > 0$, so that

$$\chi(M,N) = \dim M \otimes_R N$$
$$= \dim R/\langle \tilde{f}, \tilde{g} \rangle.$$

Observations:

- $\tilde{f}, \tilde{g}$ form a **regular sequence** in the ring R.
- $\langle s^2, st, t^2 \rangle$ is not generated by a regular sequence.

Hilbert-Samuel Definition

Let R, m and $k = R/m$ be as above and let M be a f.g. R-module. For $\ell \gg 0$, the Hilbert polynomial implies that

$$\dim_k(M/m^{\ell+1}M) = \frac{e}{d!}\ell^d + \dots$$

where $d = \dim R$ and $e = e(M)$ is the **multiplicity** of M.

Theorem: If $\dim R = 0$, then

$$e(R) = \dim_k R.$$

Refinement:

Let I be an ideal with $m^s M \subset IM$ for some s. Then $\ell \gg 0$ implies that

$$\dim_k(M/I^{\ell+1}M) = \frac{\tilde{e}}{d!}\ell^d + \dots$$

$\tilde{e} = e(I, M)$ is the **multiplicity** of I in M.

Main Claim:

In the Degree Formula,

$$m(p) = e(I_p, R_p),$$

for $R_p = \mathcal{O}_{\mathbf{P}^2,p}$ and $I_p = \langle \tilde{a}, \dots, \tilde{d} \rangle$.

Combinatorial Computation

Let $I \subset k[x_1, \ldots, x_n]_{\langle x_1, \ldots, x_n \rangle} = R$ have finite codimension. Then:

- The exponents of a monomial basis of R/I give a finite set $E \subset \mathbf{Z}^n_{\geq 0}$.
- Let $C = \mathrm{Conv}(\mathbf{Z}^n_{\geq 0} \setminus E)$.

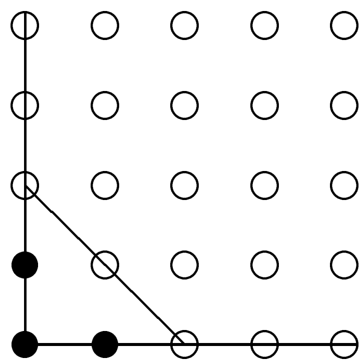
This gives the multiplicity:

Theorem:

$$e(I, R) = n!\, \mathrm{Vol}_n(\mathbf{R}^n_{\geq 0} \setminus C)$$

Example:

$I = \langle s^2, st, t^2 \rangle \subset k[s,t]_{\langle s,t \rangle}$ has basis $1, s, t$ of R/I. Then



gives $e(I, R) = 2 \cdot$ shaded area $=$ $2 \cdot 2 = 4$.

Example:

$I = \langle s^2, t^2 \rangle \subset k[s,t]_{\langle s,t \rangle}$ has the same multiplicity.

Algebraic Computation

Assume $m^s \subset I \subset R$, where $R =$ regular local ring of dimension n. Then:

- If $m^s \subset J \subset I \subset R$, then

$$e(J, R) \geq e(I, R).$$

- If in addition $I^\ell J = I^{\ell+1}$, then

$$e(J, R) = e(I, R).$$

J is a **reduction ideal** of I.

- If I is generated by a regular sequence, then

$$e(I,R) = \dim_k R/I$$

  I is a **complete intersection**.

Two Important Facts:

- I has a reduction ideal which is generated by a reg. seq.
- The reg. seq. can be chosen to be generic linear combinations of the generators of I.

Consequence of First Fact:

$$e(I, R) = \min \dim_k R/J,$$

where the minimum is taken over all complete intersection ideals J contained in I.

This follows because

$$e(I, R) \leq e(J, R) = \dim_k R/J$$

holds for any CI ideal contained in I and because (by the first fact) I has a CI reduction ideal.

Proof of Degree Formula

For $\varphi : \mathbf{P}^2 \dashrightarrow \mathbf{P}^3$ given by a, b, c, d, we need to prove

$$n^2 = \deg S \cdot \deg \varphi + \sum_{p \in Z} e(I_p, R_p)$$

for $R_p = \mathcal{O}_{\mathbf{P}^2,p}$ and $I_p = \langle \tilde{a}, \ldots, \tilde{d} \rangle$.

Pick a generic line $\ell \subset \mathbf{P}^2$. Then

$$\deg S = \# S \cap \ell.$$

We can assume:

- ℓ meets S at smooth points.
- ℓ meets S transversely.
- φ is étale above these points.

For coordinates x, y, z, w on $\mathbf{P}^3$, let

$$\ell = H_1 \cap H_2$$

for $H_1 : \alpha_1 x + \cdots + \alpha_4 w = 0$ and $H_2 : \beta_1 x + \cdots + \beta_4 w = 0$. Then, on $\mathbf{P}^2$, consider the curves

$$C : f = \alpha_1 a + \cdots + \alpha_4 d = 0$$
$$D : g = \beta_1 a + \cdots + \beta_4 d = 0.$$

Since Z is the basepoint locus of a, b, c, d, we have

$$C \cap D = \varphi^{-1}(S \cap \ell) \cup Z.$$

By Bézout's Theorem,

$$n^2 = \deg S \cdot \deg \varphi + \sum_{p \in Z} \dim_k \mathcal{O}_{\mathbf{P}^2,p}/\langle \tilde{f}, \tilde{g} \rangle.$$

However, the second important fact implies that $\tilde{f}, \tilde{g}$ generate a reduction ideal for I_p. Thus

$$e(I_p, R_p) = \dim_k \mathcal{O}_{\mathbf{P}^2,p}/\langle \tilde{f}, \tilde{g} \rangle$$

and the theorem is proved!

Another Proof

We will use Fulton's *Intersection Theory*. By p. 79, the **Segre class** of $Z \subset \mathbf{P}^2$ is the 0-cycle

$$s(Z, \mathbf{P}^2) = \sum_{p \in Z} e(I_p, R_p)[p].$$

Then Prop. 4.4 implies

$$\deg S \cdot \deg \varphi = \int_{\mathbf{P}^2} c_1(L)^2 - \int_Z (1 + c_1(L))^2 \cap s(Z, \mathbf{P}^2)$$

where $L = \mathcal{O}_{\mathbf{P}^2}(n)$. Then we are done since Z has dimension 0!

The Rees Ring

This is the graded ring

$$R_+(I) = \bigoplus_{i=0}^{\infty} I^i t^i$$

Then set

$$\widetilde{R} = R_+(I)/mR_+(I).$$

This is graded and finitely generated over $k = R/m$. One can show that $\dim \widetilde{R} = \dim R$, which we denote n.

By graded Noether normalization, there are generic $\tilde{s}_1, \ldots, \tilde{s}_n \in I/mI$ such that $\widetilde{R}$ is a f. g. module over $k[\tilde{s}_1, \ldots, \tilde{s}_n]$.

One can show:

- $R_+(I)$ is finitely generated over $R[s_1, \ldots, s_n]$.
- $s_1, \ldots, s_n$ are a regular sequence.

We may assume that the s_i are generic linear combinations of generators.

Now suppose that $u_1, \ldots, u_N$ generate $R_+(I)$ over $R[s_1, \ldots, s_n]$. Set

$$\ell = \max \text{degree of } u_1, \ldots, u_N.$$

Then one can easily show that

$$I^{\ell+1} = \langle s_1, \ldots, s_n \rangle I^{\ell}.$$

See Section 4.5 of *Cohen-Macaulay Rings* by Bruns and Herzog for details.