LECTURES ON TORIC VARIETIES

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Introduction

Toric varieties are geometric objects defined by combinatorial information. They provide a wonderful introduction to algebraic geometry and commutative algebra. In these lectures, we will explore various aspects of this rich subject. Here is a brief synopsis of the lectures:

Lecture 1: Toric Varieties, Lattices, and Cones

The first lecture will define toric varieties and give some examples. I will introduce two lattices that play an important role, the lattice of characters and the lattice of one-parameter subgroups. Examples will be given to show how these lattices arise naturally when considering toric varieties. We will also describe affine toric varieties in terms of cones and their duals.

Lecture 2: The Toric Variety of a Fan

The second lecture will discuss certain collections of cones called fans and give the classic construction of the toric variety of a fan. The basic idea is that the combinatorial data of the fan tells us how to glue together the affine toric varieties coming from the cones of the fan. We will then consider standard properties of toric varieties, including smoothness and completeness. We will also discuss normality and Cohen-Macaulayness and say a few words about simplicial fans and finite quotient singularities.

Lecture 3: Homogeneous Coordinates and Toric Ideals

The construction of a toric variety from a fan goes back to the introduction of toric varieties in the 1970s. More recently, two new constructions of toric varieties have been given. The first generalizes the construction of projective space as the quotient of affine space minus the origin. We will see how this gives explicit pictures of some toric varieties. The second construction uses monomial maps and generalizes the idea of a monomial curve, such as the twisted cubic given by $t \to (t, t^2, t^3)$. This will lead to the notions of toric ideals and non-normal toric varieties.

Lecture 4: Polytopes and Toric Varieties

In this lecture, we will see that every lattice polytope gives a projective toric variety. We will explain this from several points of view, including normal fans, monomial maps, and homogeneous coordinates. We will also briefly discuss the Dehn-Sommerville equations and Ehrhart polynomials.

Lecture 5: Toric Regularity

In the final lecture, we will define regularity in the toric context and work out its explicit meaning for $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}(q_0, \ldots, q_n)$. We will see that multi-graded commutative algebra has some subtle features.

We also include a brief summary of the background material on affine and projective varieties required for the lectures.

0. Background

Our review of affine and projective varieties assumes familiarity with Chapter I of Hartshorne's *Algebraic Geometry* [21], though we will use slightly different terminology and notation. Other introductions to algebraic geometry include [9, 20, 29].

For simplicity, we will work over the field of complex numbers $\mathbb{C}$.

0.1. **Affine Varieties.** Given polynomials $f_1, \dots, f_s \in \mathbb{C}[x_1, \dots, x_n]$, we get the *affine variety*

$$\mathbf{V}(f_1, \dots, f_s) = \{a \in \mathbb{C}^n \mid f_1(a) = \cdots = f_s(a) = 0\}.$$

More generally, if $I \subset \mathbb{C}[x_1, \dots, x_n]$ is an ideal, then we define

$$\mathbf{V}(I) = \{a \in \mathbb{C}^n \mid f(a) = 0 \text{ for all } f \in I\}.$$

If $I = \langle f_1, \dots, f_s \rangle$ is the ideal generated by $f_1, \dots, f_s$, then $\mathbf{V}(I) = \mathbf{V}(f_1, \dots, f_s)$. All ideals in $\mathbb{C}[x_1, \dots, x_n]$ are of this form by the Hilbert Basis Theorem.

Given an affine variety $V \subset \mathbb{C}^n$, we get the ideal

$$\mathbf{I}(V) = \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f(a) = 0 \text{ for all } a \in V\}.$$

If $V \subset \mathbb{C}^n$ is an affine variety, then we always have $V = \mathbf{V}(\mathbf{I}(V))$. On the other hand, if $I \subset \mathbb{C}[x_1, \dots, x_n]$ is an ideal, then the Hilbert Nullstellensatz tells us that $\sqrt{I} = \mathbf{I}(\mathbf{V}(I))$, where

$$\sqrt{I} = \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f^m \in I \text{ for some } m \geq 1\}$$

is the *radical* of I. The same result holds over any algebraically closed field.

Using the Nullstellensatz, one sees that every maximal ideal of $\mathbb{C}[x_1, \dots, x_n]$ is of the form $\langle x_1 - a_1, \dots, x_n - a_n \rangle$, where $a_i \in \mathbb{C}$. Thus there is a one-to-one correspondence between points of $\mathbb{C}^n$ and maximal ideals of $\mathbb{C}[x_1, \dots, x_n]$. This correspondence extends to a one-to-one correspondence

$$\text{affine varieties of } \mathbb{C}^n \longleftrightarrow \text{radical ideals of } \mathbb{C}[x_1, \dots, x_n].$$

Recall that an ideal I is *radical* if $I = \sqrt{I}$.

0.2. **Coordinate Rings.** Polynomials $f, g \in \mathbb{C}[x_1, \dots, x_n]$ give the same function on an affine variety V if and only if their difference lies in $\mathbf{I}(V)$. Thus the ring of such functions is isomorphic to the quotient ring

$$\mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_n]/\mathbf{I}(V).$$

This is the *coordinate ring* of V. We observe that there is a one-to-one correspondence between points of V and maximal ideals of $\mathbb{C}[V]$.

Affine varieties $V_1 \subset \mathbb{C}^n$ and $V_2 \subset \mathbb{C}^m$ are *isomorphic* if there are polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^m$ and $G : \mathbb{C}^m \to \mathbb{C}^n$ such that $F(V_1) = V_2$, $G(V_2) = V_1$, and the compositions $F \circ G$ and $G \circ F$ are the identity when restricted to V_2 and V_1 respectively. Two affine varieties are isomorphic if and only if their coordinate rings are isomorphic $\mathbb{C}$-algebras.

We characterize coordinate rings of affine varieties as follows.

Proposition 0.1. *A $\mathbb{C}$-algebra R is isomorphic to the coordinate ring of an affine variety if and only if R is a finitely generated $\mathbb{C}$-algebra with no nonzero nilpotents, i.e., if $f \in R$ satisfies $f^m = 0$, then $f = 0$.*

To emphasize the close relation between V and $\mathbb{C}[V]$, we will sometimes write

$$V = \mathrm{Spec}(\mathbb{C}[V])$$

This can be made canonical by identifying V with the set of maximal ideals of $\mathbb{C}[V]$. This is part of a general contruction in algebraic geometry which takes any commutative ring R and defines the *affine scheme* $\mathrm{Spec}(R)$. The general definition of Spec uses all prime ideals of R and not just the maximal ideals as we have done. Thus the above should be written $V = \mathrm{Specm}(\mathbb{C}[V])$, the maximal spectrum of $\mathbb{C}[V]$. Readers wishing to learn more about schemes should consult [15, 21].

Given a $\mathbb{C}$-algebra R as in Proposition 0.1, the affine variety $V = \mathrm{Spec}(R)$ can be constructed as follows. Since R is a finitely generated $\mathbb{C}$-algebra, we have an isomorphism

$$R \simeq \mathbb{C}[x_1, \ldots, x_n]/I,$$

and since R has no nilpotents, the ideal I is radical. Then $\mathrm{Spec}(R)$ can be identified with $\mathbf{V}(I) \subset \mathbb{C}^n$.

0.3. **The Zariski Topology.** Given an affine variety $V \subset \mathbb{C}^n$, a subset $W \subset V$ is a *subvariety* if W is also an affine variety. An affine variety has two interesting topologies. First, we have the induced topology from the usual topology on $\mathbb{C}^n$. This is sometimes called the *classical topology.* The other topology is defined as follows. Given a subvariety $W \subset V$, the complement $V \setminus W$ is called a *Zariski open subset* of V. The Zariski open subsets of V form a topology on V, called the *Zariski topology.* Since every subvariety of V is closed in the classical topology (polynomials are continuous), it follows that every Zariski open subset is also open in the classical topology.

Given a subset $S \subset V$, its closure $\overline{S}$ in the Zariski topology is the smallest subvariety of V containing S. We call $\overline{S}$ the *Zariski closure* of S. Note that the Zariski closure of $S \subset \mathbb{C}^n$ is $\overline{S} = \mathbf{V}(\mathbf{I}(S))$. The Zariski closure can be strictly bigger than the closure in the classical topology.

Finally, we remark that some Zariski open subsets of an affine variety V are themselves affine varieties. Given $f \in \mathbb{C}[V] \setminus \{0\}$, let $V_f = \{a \in V \mid f(a) \neq 0\} \subset V$.

Lemma 0.2. *The Zariski open subset $V_f \subset V$ has the natural structure of an affine variety.*

Proof. Suppose $V \subset \mathbb{C}^n$ and $\mathbf{I}(V) = \langle f_1, \ldots, f_s \rangle$. Also pick $g \in \mathbb{C}[x_1, \ldots, x_n]$ so that $f = g + \mathbf{I}(V)$. Then $V_f = V \setminus \mathbf{V}(f_1, \ldots, f_s, g)$. Let $W = \mathbf{V}(f_1, \ldots, f_s, 1 - gy) \subset \mathbb{C}^n \times \mathbb{C}$, where y is a new variable. Then the projection map $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ maps W bijectively onto V_f, so that we can identify V_f with the affine variety $W \subset \mathbb{C}^n \times \mathbb{C}$. □

0.4. **Irreducible Affine Varieties.** An affine variety V is *irreducible* if it cannot be written as union of subvarieties $V = V_1 \cup V_2$ where $V_i \neq V$. Note that V is irreducible $\Leftrightarrow \mathbf{I}(V) \subset \mathbb{C}[x_1, \ldots, x_n]$ is a prime ideal $\Leftrightarrow \mathbb{C}[V]$ is an integral domain. Every affine variety V can be written as a union

$$V = V_1 \cup \cdots \cup V_r,$$

where each V_i is irreducible and $V_i \not\subset \bigcup_{j \neq i} V_j$. We call $V_1, \ldots, V_r$ the *irreducible components* of V.

When V is irreducible, the integral domain $\mathbb{C}[V]$ has a field of fractions denoted $\mathbb{C}(V)$. This is the *field of rational functions* on V. For example, when $V = \mathbb{C}^n$, $\mathbb{C}[V]$ is the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{C}(V)$ is the field of rational functions $\mathbb{C}(x_1, \ldots, x_n)$. In general, given $f/g \in \mathbb{C}(V)$, the equation $g = 0$ defines a proper subvariety $W \subset V$ and $f/g : V \setminus W \to \mathbb{C}$ is a well-defined function. This is written $f/g : V \dashrightarrow \mathbb{C}$ and is called a *rational function* on V.

When V is irreducible and $f \in \mathbb{C}[V]$ is nonzero, the *localization* of $\mathbb{C}[V]$ at f is

$$\mathbb{C}[V]_f = \{g/f^n \in \mathbb{C}(V) \mid g \in \mathbb{C}[V],\ n \geq 0\}.$$

Then $\mathrm{Spec}(\mathbb{C}[V]_f)$ is the affine variety V_f from Lemma 0.2.

In [21], $\mathbf{V}(I) \subset \mathbb{C}^n$ is denoted $Z(I) \subset \mathbb{A}^n_{\mathbb{C}}$ and called an "algebraic set," and the term "affine variety" is used only when $\mathbf{V}(I)$ is irreducible. This conflict of terminology won't cause trouble since our main objects of interest are toric varieties, which are by definition irreducible.

0.5. **Normal Affine Varieties.** Let R be an integral domain with field of fractions K. Then R is *integrally closed* if every element of K which is integral over R (meaning that it is a root of a monic polynomial in $R[x]$) actually lies in R.

Let V be an irreducible affine variety, so that $\mathbb{C}[V]$ is an integral domain. Then V is *normal* if $\mathbb{C}[V]$ is integrally closed. For example, $\mathbb{C}^n$ is normal since its coordinate ring $\mathbb{C}[x_1, \ldots, x_n]$ is a UFD and hence is integrally closed.

A classic example of a non-normal variety is $C = \mathbf{V}(x^3 - y^2) \subset \mathbb{C}^2$. This is an irreducible plane curve with a cusp at the origin, and $\mathbb{C}[C] = \mathbb{C}[x, y]/\langle x^3 - y^2 \rangle$. However, if X and Y are the cosets of x and y in $\mathbb{C}[C]$ respectively, then $Y/X \in \mathbb{C}(C) \setminus \mathbb{C}[C]$ satisfies $(Y/X)^2 = X$. This implies that $\mathbb{C}[C]$ is not integrally closed. We will see in LECTURE 1 that C is a non-normal toric variety.

Another example is $V = \mathbf{V}(xy - zw) \subset \mathbb{C}^4$. It is not obvious, but V is normal. This will follow from the description

$$\mathbb{C}[V] \simeq \mathbb{C}[ab, cd, ac, bd] \subset \mathbb{C}[a, b, c, d]$$

to be given in the lectures. The ring $\mathbb{C}[ab, cd, ac, bd]$ is an example of a *semigroup algebra*. For this ring, we will see that normality follows from a property called *saturation*.

For us, normality is crucial because the nicest toric varieties are normal. We will also discuss non-normal toric varieties, but the strongest results hold only in the normal case.

Finally, any irreducible affine variety V has a *normalization*. To define this, first consider

$$\mathbb{C}[V]' = \{\alpha \in \mathbb{C}(V) \mid \alpha \text{ is integral over } \mathbb{C}[V]\}.$$

We call $\mathbb{C}[V]'$ the *integral closure* of $\mathbb{C}[V]$. It is easy to see that $\mathbb{C}[V]'$ is integrally closed. With more work, one can also show that $\mathbb{C}[V]'$ is a finitely generated $\mathbb{C}$-algebra. This gives the normal affine variety

$$V' = \mathrm{Spec}(\mathbb{C}[V]')$$

which is the *normalization* of V. Note that the natural inclusion $\mathbb{C}[V] \subset \mathbb{C}[V]' = \mathbb{C}[V']$ corresponds to a map $V' \to V$. This is called the *normalization map*.

0.6. **Projective Space.** We define *n-dimensional projective space* to be the set

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\!\sim,$$

where $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n) \iff$ there is $\lambda \in \mathbb{C}^*$ with $(a_0, \ldots, a_n) = \lambda(b_0, \ldots, b_n)$. Here, we use $\mathbb{C}^*$ to denote $\mathbb{C} \setminus \{0\}$, which is a group under multiplication. As we vary $\lambda \in \mathbb{C}^*$, the points $\lambda(b_0, \ldots, b_n)$ lie on a line through the origin. Thus we get a bijection

$$\mathbb{P}^n \simeq \{\text{lines through the origin in } \mathbb{C}^{n+1}\}.$$

A point p of $\mathbb{P}^n$ will be written $(a_0, \ldots, a_n)$. This is only unique up to multiplication by elements of $\mathbb{C}^*$. We call $(a_0, \ldots, a_n)$ *homogeneous coordinates* of p. This is sometimes written $p = [a_0, \ldots, a_n]$ or $p = (a_0 : \cdots : a_n)$. We prefer to write $p = (a_0, \ldots, a_n)$, where it will be clear from the context that we are using homogeneous coordinates.

0.7. **Homogeneous Ideals and Projective Varieties.** A polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ is *homogeneous of degree d* if every term of f has total degree d. This is equivalent to the identity

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n), \quad \lambda \in \mathbb{C}^*.$$

Any $f \in \mathbb{C}[x_0, \ldots, x_n]$ can be written uniquely in the form $f = \sum_{d \geq 0} f_d$, where f_d is homogeneous of degree d. We call f_d the *homogeneous components* of f.

Now let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be homogeneous of degree d. Given $p \in \mathbb{P}^n$, we can't define "$f(p)$" since homogeneous coordinates aren't well-defined. However, the equation $f(p) = 0$ is well-defined. Thus, homogeneous polynomials $f_1, \ldots, f_s \in \mathbb{C}[x_0, \ldots, x_n]$ define the *projective variety*

$$\mathbf{V}(f_1, \ldots, f_s) = \{a \in \mathbb{P}^n \mid f_1(a) = \cdots = f_s(a) = 0\} \subset \mathbb{P}^n.$$

It is clear what we mean by a *subvariety* W of a projective variety $V \subset \mathbb{P}^n$. Then $V \setminus W$ is *Zariski open* subset of V. This gives the *Zariski topology* on V, and then *Zariski closure* is defined in the obvious way. We can also define what it means for a projective variety to be *irreducible.*

An ideal $I \subset \mathbb{C}[x_0, \ldots, x_n]$ is *homogeneous* if it is generated by homogeneous polynomials. Such an ideal I defines the projective variety

$$\mathbf{V}(I) = \{a \in \mathbb{C}^n \mid f(a) = 0 \text{ for all } f \in I\}.$$

Conversely, given an projective variety $V \subset \mathbb{C}^n$, we get the homogeneous ideal

$$\mathbf{I}(V) = \{f \in \mathbb{C}[x_0, \ldots, x_n] \mid f(a) = 0 \text{ for all } a \in V\}.$$

We call $\langle x_0, \ldots, x_n \rangle \subset \mathbb{C}[x_0, \ldots, x_n]$ the *irrelevant ideal.* It is easy to see that $\mathbf{V}(I) = \emptyset$ when I contains a power of the irrelevant ideal. This part of the projective version of the Projective Nullstellensatz, which states that if $I \subset \mathbb{C}[x_0, \ldots, x_n]$ is a homogeneous ideal, then

$$\mathbf{V}(I) = \emptyset \iff \langle x_0, \ldots, x_n \rangle^m \subset I \text{ for some } m \geq 0$$

and that

$$\mathbf{V}(I) \neq \emptyset \Longrightarrow \mathbf{I}(\mathbf{V}(I)) = \sqrt{I}.$$

0.8. **Rational Functions.** A homogeneous polynomial in $\mathbb{C}[x_0, \ldots, x_n]$ does not give a function on $\mathbb{P}^n$. However, the quotient of two such polynomials works, provided they have the same degree. More precisely, suppose that $f, g \in \mathbb{C}[x_0, \ldots, x_n]$ have degree d and that $g \neq 0$. Then we get a well-defined function

$$\frac{f}{g} : \mathbb{P}^n \setminus \mathbf{V}(g) \longrightarrow \mathbb{C}$$

We write this as $f/g : \mathbb{P}^n \dashrightarrow \mathbb{C}$ and say that f/g is a *rational function* on $\mathbb{P}^n$. The field of all rational functions on $\mathbb{P}^n$ is denoted $\mathbb{C}(\mathbb{P}^n)$. Similarly, if $V \subset \mathbb{P}^n$ is irreducible, then we get the field $\mathbb{C}(V)$ of rational functions on V.

0.9. **Mappings Between Projective Varieties.** Let $V \subset \mathbb{P}^n$ is a projective variety. We say that homogeneous polynomials $f_0, \ldots, f_m \in \mathbb{C}[x_0, \ldots, x_n]$ of the same degree have no *base points* on V if $V \cap \mathbf{V}(f_0, \ldots, f_m) = \emptyset$. When this happens, the map

$$(a_0, \ldots, a_n) \mapsto (f_0(a_0, \ldots, a_n), \ldots, f_m(a_0, \ldots, a_n))$$

induces a well-defined function $F : V \longrightarrow \mathbb{P}^m$. An important fact is that in this situation, the image $F(V) \subset \mathbb{P}^m$ is a projective variety.

In contrast, when V is affine and $F : V \to \mathbb{C}^m$ is a polynomial map, the image $F(V) \subset \mathbb{C}^m$ need not be a variety. For example, if V is $\mathbf{V}(xy - 1) \subset \mathbb{C}^2$ and F is $F(x, y) = x$, then $F(V)$ is not a variety. The fact that $F(V)$ is a variety in the projective case is one reason why projective varieties are so useful in algebraic geometry.

0.10. **Affine Open Subsets.** $\mathbb{P}^n$ contains copies of the affine space $\mathbb{C}^n$ as follows. For $0 \leq i \leq n$, consider the Zariski open set $U_i = \mathbb{P}^n \setminus \mathbf{V}(x_i)$. It is straightforward to show that $U_i \simeq \mathbb{C}^n$ via $(a_0, \ldots, a_n) \mapsto (a_0/a_i, \ldots, a_{i-1}/a_i, a_{i+1}/a_i, \ldots, a_n/a_i)$ and that $\mathbf{V}(x_i) \simeq \mathbb{P}^{n-1}$ via $(a_0, \ldots, a_n) \mapsto (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$. Thus we can regard $\mathbb{P}^n$ as $\mathbb{C}^n$ together with a copy of $\mathbb{P}^{n-1}$ "at infinity". Furthermore, $\mathbb{P}^n = U_0 \cup \cdots \cup U_n$, so that projective space is a union of affine spaces.

More generally, let $V = \mathbf{V}(f_1, \ldots, f_s) \subset \mathbb{P}^n$ be a projective variety. Then, under the above map $U_i \simeq \mathbb{C}^n$, $V \cap U_i$ corresponds to an affine variety defined by the vanishing of $\tilde{f}_j$, where

$$\tilde{f}_j(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = f_j(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n).$$

We call $\tilde{f}_j$ the *dehomogenization* of f_j with respect to x_i. Then

$$V = (V \cap U_0) \cup \cdots \cup (V \cap U_n)$$

shows that every projective variety can be regarded as a union of affine varieties. This allows us to define what it means for a projective variety to be normal.

Another way to think about $U_i \simeq \mathbb{C}^n$ is to use $x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i$ as variables on $\mathbb{C}^n$, i.e.,

$$U_i = \operatorname{Spec}(\mathbb{C}[x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i]). \tag{0.1}$$

Then the dehomogenization map is $f \mapsto f/x_i^d$, where $f \in \mathbb{C}[x_0, \ldots, x_n]$ is homogeneous of degree d. This approach preserves rational functions since $f/g \mapsto (f/x_i^d)/(g/x_i^d)$ induces an isomorphism

$$\mathbb{C}(\mathbb{P}^n) \simeq \mathbb{C}(x_0/x_i, \ldots, x_{i-1}/x_i, x_{i-1}/x_i, \ldots, x_n/x_i).$$

0.11. **Weighted Projective Space.** Given positive integers $q_0, \ldots, q_n$ with $\gcd(q_0, \ldots, q_n) = 1$, we get the *weighted projective space*

$$\mathbb{P}(q_0, \ldots, q_n) = (\mathbb{C}^{n+1} - \{0\})/\sim,$$

where $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n) \iff$ there is $\lambda \in \mathbb{C}^*$ with $(a_0, \ldots, a_n) = (\lambda^{q_0} b_0, \ldots, \lambda^{q_n} b_n)$. Obviously $\mathbb{P}(1, \ldots, 1) = \mathbb{P}^n$. We call $q_0, \ldots, q_n$ the *weights* of the weighted projective space.

In terms of the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$, this means that x_i has degree q_i, and a polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ is *weighted homogeneous* of (weighted) degree d if

$$f(\lambda^{q_0} x_0, \ldots, \lambda^{q_n} x_n) = \lambda^d f(x_0, \ldots, x_n).$$

These polynomials enable us to define *weighted homogeneous ideals* $I \subset \mathbb{C}[x_0, \ldots, x_n]$ and the corresponding *weighted projective varieties* $\mathbf{V}(I) \subset \mathbb{P}(q_0, \ldots, q_n)$.

In LECTURE 1 we will see that $\mathbb{P}(q_0, \ldots, q_n)$ is a toric variety. Here are two ways to think about the weighted projective plane $\mathbb{P}(1,1,2)$.

Exercise 0.1. Consider $\mathbb{P}(1,1,2)$ with variables x_0, x_1, x_2 of degrees $1, 1, 2$ respectively.

a. Show that $x_0^2, x_0x_1, x_1^2, x_2$ are (weighted) homogeneous of degree 2.
b. Show that $(a_0, a_1, a_2) \mapsto (a_0^2, a_0a_1, a_1^2, a_2)$ is a well-defined map $F : \mathbb{P}(1,1,2) \to \mathbb{P}^3$.
c. Show that the map F of part b is injective and that its image is the surface defined by the equation $y_0y_2 - y_1^2 = 0$, where y_0, y_1, y_2, y_3 are the coordinates of $\mathbb{P}^3$.

Exercise 0.2. Show that $(a_0, b_0, c_0) \to (a_0, b_0, c_0^2)$ gives a well-defined map $\mathbb{P}^2 \to \mathbb{P}(1,1,2)$. Also show that this map is surjective and is two-to-one except above $(0,0,1) \in \mathbb{P}(1,1,2)$.

More generally, one can show that $\mathbb{P}(q_0, \ldots, q_n)$ can always be embedded into projective space. We can also cover $\mathbb{P}(q_0, \ldots, q_n)$ by affine open subsets, similar to $\mathbb{P}^n$. Rather than work this out in general, we will restrict to the case of $\mathbb{P}(1,1,2)$.

Exercise 0.3. Let $U_i = \{(a_0, a_1, a_2) \in \mathbb{P}(1,1,2) \mid a_i \neq 0\}$. Note that $\mathbb{P}(1,1,2) = U_0 \cup U_1 \cup U_2$.

a. Show that $U_0 \simeq \mathbb{C}^2$ via $(a, b, c) \mapsto (b/a, c/a^2)$ and $U_1 \simeq \mathbb{C}^2$ via $(a, b, c) \mapsto (a/b, c/b^2)$.
b. Let $V = \mathbf{V}(xz - y^2) \subset \mathbb{C}^3$. Show that $U_2 \simeq V$ via $(a, b, c) \mapsto (a^2/c, ab/c, b^2/c)$.

0.12. **Abstract Varieties.** Projective varieties can be expressed as unions of affine varieties, and the same is true for weighted projective varieties. More generally, one can define *abstract varieties* to be unions affine varieties. The full definition requires the study of *affine schemes*, *sheaves*, and *ringed spaces*—see [15, Section 1.1] and [21, II.1 and II.2] for the details. We will use abstract varieties in LECTURE 2 when we construct the toric variety of a fan.

Informally, one can define an abstract variety as follows. Suppose that we have a collection

$$(\{V_\alpha\}_\alpha, \{V_{\alpha\beta}\}_{\alpha,\beta}, \{g_{\alpha\beta}\}_{\alpha,\beta}),$$

where V_α is an affine variety, $V_{\alpha\beta} \subset V_\alpha$ is Zariski open, and the $g_{\alpha\beta} : V_{\alpha\beta} \simeq V_{\beta\alpha}$ are isomorphisms that satisfy the compatibility conditions

- $g_{\alpha\alpha} = 1_{V_\alpha}$ for every α.
- $g_{\beta\gamma}\big|_{V_{\beta\alpha}\cap V_{\beta\gamma}} \circ g_{\alpha\beta}\big|_{V_{\alpha\beta}\cap V_{\alpha\gamma}} = g_{\alpha\gamma}\big|_{V_{\alpha\beta}\cap V_{\alpha\gamma}}$ for every α, β, γ.

Then we get the abstract variety

$$X = \bigcup_\alpha V_\alpha / \sim$$

where $a \in V_\alpha$ is equivalent to $b \in V_\beta$ if $a \in V_{\alpha\beta}$ and $b = g_{\alpha\beta}(a)$. We say that X is obtained by *gluing together* the V_α along the $V_{\alpha\beta}$ via the $g_{\alpha\beta}$.

Example 0.1. Consider the variety X constructed by identifying two copies of $\mathbb{C}$ along $\mathbb{C}^*$. In the above notation, this corresponds to $V_0 = V_1 = \mathbb{C}$, $V_{01} = V_{10} = \mathbb{C}^*$, and $g_{01}(x) = x$. The abstract variety X lookes like $\mathbb{C}$ except that it has two copies of the origin.

On the other hand, if we take V_0, V_1, V_{01}, V_{10} as above and let $g_{01}(x) = x^{-1}$, then the resulting abstract variety is $\mathbb{P}^1$. □

The variety X constructed in this example is not Hausdorff in the classical topology. The algebraic name for this condition is *separated*, to be defined below. With the exception of Example 0.1, all varieties considered in these lectures will be separated.

0.13. **Cartesian Products.** Suppose that we have affine varieties $V = \mathbf{V}(f_1, \ldots, f_s) \subset \mathbb{C}^n$, with variables $x_1, \ldots, x_n$, and $W = \mathbf{V}(g_1, \ldots, g_t) \subset \mathbb{C}^m$, with variables $y_1, \ldots, y_m$. Then

$$V \times W \subset \mathbb{C}^n \times \mathbb{C}^m = \mathbb{C}^{n+m}$$

is $\mathbf{V}(f_1, \ldots, f_s, g_1, \ldots, g_t)$, where $f_i(x_1, \ldots, x_n), g_j(y_1, \ldots, y_m) \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. We call $V \times W$ the *Cartesian product* of V and W. The coordinate ring of $V \times W$ is $\mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}[W]$.

More generally, one can define $V \times W$ when V and W are projective, weighted projective, or abstract varieties. We also note that an abstract variety V is separated if and only if the diagonal map $V \to V \times V$ has closed image. Complete discussions of Cartesian products and separated schemes can be found in [15, 21].

0.14. **The Local Ring of an Irreducible Divisor.** Let X be an irreducible variety with function field $\mathbb{C}(X)$. An irreducible subvariety $Y \subset X$ is an *irreducible divisor* if Y has codimension 1 in X. Let $Y \subset X$ be an irreducible divisor and set

$$\mathcal{O}_{X,Y} = \{f \in \mathbb{C}(X) \mid f \text{ is defined on a nonempty Zariski open subset of } Y\}.$$

Recall that every $f \in \mathbb{C}(X)$ is defined on some nonempty Zariski open $U \subset X$. Then $f \in \mathcal{O}_{X,Y}$ when we can find such a U satisfying $U \cap Y \neq 0$. One can show without difficulty that $\mathcal{O}_{X,Y}$ is a local ring and that the maximal ideal consists of those $f \in \mathcal{O}_{X,Y}$ which vanish on Y.

Exercise 0.4. Let $Y = \mathbf{V}(x) \subset \mathbb{C}^2$ be the y-axis.

a. Prove that

$$\mathcal{O}_{\mathbb{C}^2,Y} = \Big\{\frac{P(x,y)}{Q(x,y)} \mid P(x,y), Q(x,y) \in \mathbb{C}[x,y],\ Q(0,y) \neq 0\Big\}.$$

b. Given $f \in \mathbb{C}(x,y)$, prove that $f = x^m g$, where $m \in \mathbb{Z}$ and $g \in \mathcal{O}_{\mathbb{C}^2,Y}$ is a unit.
c. Prove that every nonzero ideal of $\mathcal{O}_{\mathbb{C}^2,Y}$ is of the form $\langle x^m \rangle$ for some $m \geq 0$.

Given $f \in \mathbb{C}(x,y)$, this exercise tells us that $f = x^m g$ for $m \in \mathbb{Z}$ and g a unit in $\mathcal{O}_{\mathbb{C}^2,Y}$. We call m the *order of vanishing* of f on $Y = \mathbf{V}(x) \subset \mathbb{C}^2$ and denote it by $\mathrm{ord}_Y(f)$.

The crucial observation is that Exercise 0.4 generalizes to any normal variety. Let R be an integral domain with field of fractions K, and set $K^* = K \setminus \{0\}$. Then R is a *discrete valuation ring* if there is a surjective function

$$\mathrm{ord}_R : K^* \to \mathbb{Z}$$

such that every for $a, b \in K^*$, we have:

- $\mathrm{ord}_R(ab) = \mathrm{ord}_R(a) + \mathrm{ord}_R(b)$.
- $\mathrm{ord}_R(a+b) \geq \min(\mathrm{ord}_R(a), \mathrm{ord}_R(b))$ provided $a + b \neq 0$.
- $R = \{a \in K^* \mid \mathrm{ord}_R(a) \geq 0\} \cup \{0\}$.

We say that ord_R is a *valuation* on K and that R is its *valuation ring*. One can show that R is a local ring with $\mathfrak{m} = \{a \in R \mid \mathrm{ord}_R(a) > 0\}$ as maximal ideal. Furthermore, if $a \in R$ satisfies $\mathrm{ord}_R(a) = 1$, then $\mathfrak{m} = \langle a \rangle$ and every nonzero ideal of R is of the form $\langle a^m \rangle$ for some $m \geq 0$. A discrete valuation ring is an integrally closed 1-dimensional Noetherian local ring.

Here are two classic examples of discrete valuation rings.

Example 0.2. Let p be prime. Then $\mathbb{Z}_{(p)} = \{a/b \mid a, b \in \mathbb{Z},\ \gcd(p,b) = 1\}$ is a discrete valuation ring. This gives the *p-adic valuation*, denoted ord_p.

Example 0.3. The ring $\mathbb{C}\{\{z\}\}$ of complex power series with positive radius of convergence is a discrete valuation ring. The valuation gives the order of vanishing of a nonzero element of $\mathbb{C}\{\{z\}\}$.

For us, the main result we need is as follows.

Theorem 0.3. *Let Y be an irreducible divisor in a normal variety X. Then $\mathcal{O}_{X,Y}$ is a discrete valuation ring.*

In down-to-earth terms, this means that in a normal variety, one can define the order of vanishing of a rational function along an irreducible divisor. In Lecture 3, we will compute the order of vanishing of a character along a torus-invariant irreducible divisor in a normal toric variety.

0.15. **Weil Divisors.** A *Weil divisor* on a normal variety X is a finite formal sum

$$D = \sum_{i=1}^{s} a_i D_i$$

where the D_i are distinct irreducible divisors of X and $a_i \in \mathbb{Z}$.

Given a nonzero rational function $f \in \mathbb{C}(X)^*$, we can define $\mathrm{ord}_Y(f)$ for every irreducible divisor $Y \subset X$. This gives a Weil divisor

$$\mathrm{div}(f) = \textstyle\sum_Y \mathrm{ord}_Y(f)\, Y$$

since there are at most finitely many divisors $Y \subset X$ such that $\mathrm{ord}_Y(f) \neq 0$.

Two Weil divisors D_1, D_2 on X are *linearly equivalent*, written $D_1 \sim D_2$, if there is $f \in \mathbb{C}(X)^*$ such that $\mathrm{div}(f) = D_1 - D_2$. Furthermore, we say that a Weil divisor D is a *principal divisor* if $D \sim 0$, i.e., $D = \mathrm{div}(f)$ for some $f \in \mathbb{C}(X)^*$. Finally, the group of Weil divisors on X modulo linear equivalence is the *Chow group* $A_{n-1}(X)$, where $n = \dim X$ and the subscript $n-1$ tells us that we are looking at equivalence classes of divisors.

In Lecture 3, we will see that the torus-invariant Weil divisors on a normal toric variety are especially easy to describe.

We need one final definition.

Definition 0.4. A Weil divisor $D = \sum_{i=1}^{s} a_i D_i$ is ***effective*** if $a_i \geq 0$ for $i = 1, \ldots, s$. This is often written $D \geq 0$.

0.16. **Cartier Divisors.** We will give a slightly non-standard treatment of Cartier divisors which works nicely on normal varieties. Let $D = \sum_{i=1}^{s} a_i D_i$ be a Weil divisor on a normal variety X. If $U \subset X$ is a nonempty Zariski open subset, then the *restriction* of D to U is the is Weil divisor

$$D\big|_U = \sum_{U \cap D_i \neq \emptyset} a_i\, U \cap D_i.$$

We now define a special class of Weil divisors.

Definition 0.5. Let D be a Weil divisor on a normal variety X.

(1) D is ***locally principal*** if there is an open cover $\{U_i\}_{i \in I}$ of X such that $D\big|_{U_i}$ is principal for every $i \in I$.
(2) D is ***Cartier*** if it is locally principal.

A principal divisor is obviously locally principal. Thus $\mathrm{div}(f)$ is Cartier for all $f \in \mathbb{C}(X)^*$. It is easy to see that a sum of Cartier divisor is Cartier and that any Weil divisor linearly equivalent to a Cartier divisor is Cartier.

Example 0.4. For an example of a Weil divisor which is not Cartier, consider the affine surface $V = \mathbf{V}(xz - y^2) \subset \mathbb{C}^3$. The x-axis $D = \mathbf{V}(y, z)$ is contained in V, so that D is a Weil divisor on V. However, one can show that D is *not* a Cartier divisor (see Example 6.11.3 in [21, II.6]). Note that V appeared earlier in Exercise 0.3. We will eventually see that V is a toric variety.

Finally, the group of Cartier divisors on X modulo linear equivalence is the *Picard group* $\mathrm{Pic}(X)$. We have a natural inclusion $\mathrm{Pic}(X) \subset A_{n-1}(X)$. One can prove that Weil and Cartier divisors coincide on smooth varieties, so that $\mathrm{Pic}(X) = A_{n-1}(X)$ when X is smooth.

We will see in LECTURE 4 that Cartier divisors arise naturally when considering the toric variety of a polytope.

LECTURE 1. TORIC VARIETIES, LATTICES, AND CONES

We define toric varieties, discuss two important lattices, and give some basic examples. We then explain how cones relate to affine toric varieties.

1.1. A Basic Definition.

We begin with the general definition of toric variety.

Definition 1.1. A ***toric variety*** is an irreducible variety X such that
(1) $(\mathbb{C}^*)^n$ is a Zariski open subset of X, and
(2) the action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on X.

Here, a "variety" can be affine, projective, or an abstract variety as defined in the BACKGROUND. We will see later that the theory of toric varieties works best when the variety is normal.

Here are the most basic examples of toric varieties.

Example 1.1. $(\mathbb{C}^*)^n$ and $\mathbb{C}^n$ are clearly toric varieties. As for $\mathbb{P}^n$, suppose that $x_0, \ldots, x_n$ are homogeneous coordinates on $\mathbb{P}^n$. The map

$$(\mathbb{C}^*)^n \longrightarrow \mathbb{P}^n$$

defined by $(t_1, \ldots, t_n) \mapsto (1, t_1, \ldots, t_n)$ allows us to identify $(\mathbb{C}^*)^n$ with the Zariski open subset $\mathbb{P}^n \setminus \mathbf{V}(x_0 x_1 \cdots x_n)$. Then setting

$$(t_1, \ldots, t_n) \cdot (a_0, a_1, \ldots, a_n) = (a_0, t_1 a_1, \ldots, t_n a_n)$$

shows that $\mathbb{P}^n$ is a toric variety. □

1.2. Two Lattices.

A *lattice* N is a free Abelian group of finite rank. Picking a $\mathbb{Z}$-basis of N gives an isomorphism $N \simeq \mathbb{Z}^n$. From N, we get the *dual lattice*

$$M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}).$$

The canonical pairing between these lattices is denoted $\langle m, u\rangle$ for $m \in M, u \in N$. Given a $\mathbb{Z}$-basis of N, the dual basis of M gives an isomorphism $M \simeq \mathbb{Z}^n$ such that $\langle m, u\rangle$ becomes dot product.

Given a lattice N, an isomorphism $N \simeq \mathbb{Z}^n$ induces an isomorphism

$$N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*)^n.$$

We call $T(N) = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ the *torus* of N. The lattices N and M relate to $T(N)$ as follows:

- First, $u \in N$ gives
$$\lambda^u : \mathbb{C}^* \to T(N)$$
defined by $\lambda^u(t) = u \otimes t$. This is a 1-*parameter subgroup* of $T(N)$. If an isomorphism $N \simeq \mathbb{Z}^n$ takes u to $(a_1, \ldots, a_n)$, then
$$\lambda^u(t) = (t^{a_1}, \ldots, t^{a_n})$$
under the induced isomorphism $T(N) \simeq (\mathbb{C}^*)^n$.
- Second, $m \in M$ gives
$$\chi^m : T(N) \to \mathbb{C}^*$$
defined by $\chi^m(\sum_{i=1}^{\ell} u_i \otimes t_i) = \prod_{i=1}^{\ell} t^{\langle m_i, u_i\rangle}$. This is a *character* of $T(N)$, so that M is its *character group*. If $M \simeq \mathbb{Z}^n$ takes m to $(b_1, \ldots, b_n)$, then
$$\chi^m(t_1, \ldots, t_n) = t_1^{b_1} \cdots t_n^{b_n}$$

under the isomorphism $T(N) \simeq (\mathbb{C}^*)^n$. We call $t_1^{b_1} \cdots t_n^{b_n}$ a *Laurent monomial* and often write t^m instead of χ^m. The monomials t^m lie in the ring $\mathbb{C}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ of *Laurent polynomials.*

In general, a toric variety consists of $T(N) \simeq (\mathbb{C}^*)^n$ plus some "extra stuff." In the case of an affine toric variety, we will see that the "extra stuff" is determined by which characters χ^m extend to functions on the variety. Here is an example.

Example 1.2. Consider the toric variety $\mathbb{C}^n$. Here, $N = M = \mathbb{Z}^n$, so that we can identify characters and Laurent monomials. Then one easily sees that the Laurent monomial $t^m = t_1^{b_1} \cdots t_n^{b_n}$ determined by $m = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ extends to a function $\mathbb{C}^n \to \mathbb{C}$ if and only if $b_i \geq 0$ for all i. Below we will construct $\mathbb{C}^n$ using these Laurent monomials. □

1.3. **Further Examples.** Besides the basic examples given by $(\mathbb{C}^*)^n$, $\mathbb{C}^n$ and $\mathbb{P}^n$, there are many more toric varieties, including the following.

Example 1.3. If X and Y are toric varieties, then so is $X \times Y$. This shows, for instance, that $\mathbb{P}^n \times \mathbb{P}^m$ is a toric variety. □

Example 1.4. The cuspidal cubic $C = \mathbf{V}(y^2 - x^3) \subset \mathbb{C}^2$ from Section 0.5 of the BACKGROUND contains $\mathbb{C}^*$ via $t \mapsto (t^2, t^3)$, and $\mathbb{C}^*$ acts on C via $t \cdot (u, v) = (t^2 u, t^3 v)$. Hence C is toric. □

The previous example is interesting because it is a non-normal toric variety. In dimension one, the only normal toric varieties are $\mathbb{C}^*$, $\mathbb{C}$ and $\mathbb{P}^1$.

Example 1.5. Consider $V = \mathbf{V}(xy - zw) \subset \mathbb{C}^4$. This contains the torus $(\mathbb{C}^*)^3$ via the map

$$(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1}).$$

Question: Which Laurent monomials t^m extend to functions $V \to \mathbb{C}$? If $m = (a, b, c) \in M = \mathbb{Z}^3$, then we get the function on V defined by $x^a y^b z^c$. If $a, b, c \geq 0$, then this certainly extends. However, suppose that $c < 0$ and $a + c, b + c \geq 0$. Then, since $xy = zw$ on V, we have

$$x^a y^b z^c = x^a y^b \left(\frac{xy}{w}\right)^c = x^{a+c} y^{b+c} w^{-c},$$

which shows that t^m extends to a function $V \to \mathbb{C}$. We will see below that the inequalities

$$a \geq 0,\ b \geq 0,\ a + c \geq 0,\ b + c \geq 0 \tag{1.1}$$

define the *dual cone* corresponding to the normal affine toric variety V. □

Example 1.6. Consider the weighted projective space $\mathbb{P}(q_0, \ldots, q_n)$, where $q_0, \ldots, q_n$ are positive integers satisfying $\gcd(q_0, \ldots, q_n) = 1$. Recall that

$$\mathbb{P}(q_0, \ldots, q_n) = (\mathbb{C}^{n+1} - \{0\})/\sim,$$

where $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n) \iff$ there is $\lambda \in \mathbb{C}^*$ with $(a_0, \ldots, a_n) = (\lambda^{q_0} b_0, \ldots, \lambda^{q_n} b_n)$. The image of $(\mathbb{C}^*)^{n+1} \subset \mathbb{C}^{n+1} - \{0\}$ in $\mathbb{P}(q_0, \ldots, q_n)$ is the quotient $(\mathbb{C}^*)^{n+1}/\mathbb{C}^*$, where we regard $\mathbb{C}^*$ as subgroup of $(\mathbb{C}^*)^{n+1}$ via the map $\lambda \mapsto (\lambda^{q_0}, \ldots, \lambda^{q_n})$. By making $(q_0, \ldots, q_n)$ the first column of a matrix $M \in \mathrm{GL}_{n+1}(\mathbb{Z})$ and using M to define an automorphism of $(\mathbb{C}^*)^{n+1}$, one sees that

$$(\mathbb{C}^*)^{n+1}/\mathbb{C}^* \simeq (\mathbb{C}^*)^n.$$

Via this isomorphism, the action of $(\mathbb{C}^*)^{n+1}$ on $\mathbb{C}^{n+1} - \{0\}$ descends to give an action of $(\mathbb{C}^*)^n$ on $\mathbb{P}(q_0, \ldots, q_n)$. This shows that $\mathbb{P}(q_0, \ldots, q_n)$ is a toric variety. □

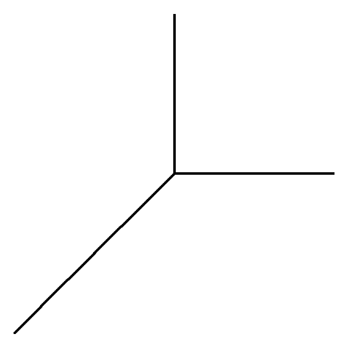
We can also cover a weighted projective space by affine open subsets, each of which is an affine toric variety. We will prove this in LECTURE 2. For now, we will restrict to the case of $\mathbb{P}(1, 1, 2)$. Here, we have the Zariski open sets $U_i = \{(a_0, a_1, a_2) \in \mathbb{P}(1, 1, 2) \mid a_i \neq 0\}$.

Exercise 1.1. Let U_0, U_1, U_2 be the subsets of $\mathbb{P}(1,1,2)$ defined above.

a. Show that $U_0 \simeq \mathbb{C}^2$ via $(a,b,c) \mapsto (b/a, c/a^2)$ and $U_1 \simeq \mathbb{C}^2$ via $(a,b,c) \mapsto (a/b, c/b^2)$.
b. Let $V = \mathbf{V}(xz - y^2) \subset \mathbb{C}^3$. Show that $U_2 \simeq V$ via $(a,b,c) \mapsto (a^2/c, ab/c, b^2/c)$.
c. Use the map $(\mathbb{C}^*)^2 \to V$ defined by $(t_1, t_2) \mapsto (t_1^2, t_1 t_2, t_2^2)$ to prove that V is an affine toric variety.

We've already seen that $\mathbb{P}^2$ is a toric variety. Here is a preview of its underlying structure.

Example 1.7. Let's show that $(\mathbb{C}^*)^2 \subset \mathbb{P}^2$ gives the following picture in $\mathbb{R}^2$:



(1.2)

A 1-parameter subgroup $u \in N = \mathbb{Z}^2$ gives a map $\lambda^u : \mathbb{C}^* \to \mathbb{P}^2$. Since $\mathbb{P}^2$ is compact, the limit $\lim_{t\to 0} \lambda^u(t)$ exists in $\mathbb{P}^2$. If $u = (a,b) \in \mathbb{Z}^2$, then

$$\lambda^u(t) = (1, t^a, t^b).$$

It is then straightforward to compute that

$$\lim_{t\to 0} \lambda^u(t) = \lim_{t\to 0}(1, t^a, t^b) = \begin{cases} (1,0,0) & a,b > 0 \\ (1,0,1) & a > 0, b = 0 \\ (1,1,0) & a = 0, b > 0 \\ (1,1,1) & a = b = 0 \\ (0,0,1) & a > b, b < 0 \\ (0,1,0) & a < 0, a < b \\ (0,1,1) & a < 0, a = b. \end{cases} \tag{1.3}$$

The first four cases are trivial. To see how the fifth case works, note that

$$\lim_{t\to 0}(1, t^a, t^b) = \lim_{t\to 0}(t^{-b}, t^{a-b}, 1)$$

since these are homogeneous coordinates. Then $a > b$ and $b < 0$ imply that the limit is $(0,0,1)$, as claimed. The last two cases are similar.

Now observe that (1.2) decomposes the plane $\mathbb{R}^2 = N \otimes_{\mathbb{Z}} \mathbb{R}$ into 7 disjoint regions:

- The open sets $a, b > 0$; $a < 0, a < b$; and $a > b, b < 0$.
- The open rays $a > 0, b = 0$; $a = 0, b > 0$; and $a < 0, a = b$.
- The point $a = b = 0$.

The corresponds perfectly with (1.3). We will see in LECTURE 2 that (1.2) is the *fan* corresponding to the toric variety $\mathbb{P}^2$. □

1.4. **Cones, Duals, and Faces.** We will follow Section 1.2 of [17], omitting proofs.

Definition 1.2. A ***convex polyhedral cone*** in $\mathbb{R}^n$ is a subset of the form

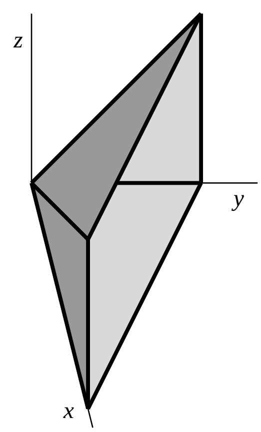
$$\sigma = \mathrm{Cone}(S) = \Big\{ \sum_{v \in S} \lambda_v v \mid \lambda_v \geq 0 \Big\} \subset \mathbb{R}^n,$$

where $S \subset \mathbb{R}^n$ is finite. We say that σ is ***generated*** by S. Also set $\mathrm{Cone}(\emptyset) = \{0\}$.

Note that σ is *convex*, meaning that $x, y \in \sigma \Rightarrow \lambda x + (1-\lambda)y \in \sigma$ for all $0 \leq \lambda \leq 1$ and is a *cone*, meaning that $x \in \sigma \Rightarrow \lambda x \in \sigma$ for all $\lambda \geq 0$. We will see below that σ is closed.

Examples of convex polyhedral cones include the first quadrant in $\mathbb{R}^2$ or first octant in $\mathbb{R}^3$. Here is a less trivial example.

Example 1.8. The cone in $\mathbb{R}^3$ generated by e_1, e_2, $e_1 + e_3$ and $e_2 + e_3$ is



We will see below that this cone is closely related to Example 1.5. □

It is also possible to have cones that contain entire lines. For example, $e_1, -e_1$ generate a cone in $\mathbb{R}^2$ which is just the x-axis, while $e_1, -e_1, e_2$ generates the closed upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. As we will see below, these last two examples are not *strongly convex*. The largest possible convex polyhedral cone is $\mathbb{R}^n$ while the smallest is the trivial cone $\{0\}$.

We can also create cones using *polytopes*, which are defined as follows.

Definition 1.3. A ***polytope*** in $\mathbb{R}^n$ is a subset of the form

$$P = \mathrm{Conv}(S) = \Big\{ \sum_{v \in S} \lambda_v v \mid \lambda_v \geq 0,\ \sum_{v \in S} \lambda_v = 1 \Big\} \subset \mathbb{R}^n,$$

where $S \subset \mathbb{R}^n$ is finite. We say that P is the ***convex hull*** of S.

Polytopes include regular n-gons in $\mathbb{R}^2$ and cubes, tetrahedra, octahedra, etc. in $\mathbb{R}^3$. We will see in Lecture 4 that polytopes play a prominent role in toric geometry. Here, however, we make the simple observation that a polytope in $\mathbb{R}^n$ gives a convex polyhedral cone in $\mathbb{R}^{n+1}$.

Exercise 1.2. Let $P = \mathrm{Conv}(S)$ be a polytope in $\mathbb{R}^n$ and regard $\mathbb{R}^n$ as the hyperplane $x_{n+1} = 1$ in $\mathbb{R}^{n+1}$. Then let

$$\sigma = \{\lambda \cdot (v, 1) \mid v \in P,\ \lambda \geq 0\}.$$

Prove that σ is a convex polyhedral cone. Hint: Consider $\mathrm{Cone}(S \times \{1\})$.

This exercise can be generalized to show that a polytope lying in an affine hyperplane not containing the origin gives a convex polyhedral cone. Be sure you can draw a picture of this.

The *dimension* of a convex polyhedral cone σ, denoted $\dim\sigma$, is defined to be the dimension of the smallest subspace $\mathbb{R}\sigma$ containing σ. We call $\mathbb{R}\sigma$ the *span* of σ. Note that $\dim\sigma = \dim\mathbb{R}\sigma$.

Exercise 1.3. The dimension of a polytope $P \subset \mathbb{R}^n$ is the dimension of the smallest affine space (= translate of a subspace) containing P. Now let $\sigma \subset \mathbb{R}^{n+1}$ be the cone determined by P as in Exercise 1.2. Prove that $\dim\sigma = \dim P + 1$. □

Given a convex polyhedral cone $\sigma \subset \mathbb{R}^n$, its *dual cone* is the set

$$\sigma^\vee = \{u \in \mathbb{R}^{n*} \mid \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\},$$

where $\mathbb{R}^{n*}$ is the dual space of $\mathbb{R}^n$ and $\langle u, v\rangle$ is the natural pairing between $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^{n*}$.

If $u \neq 0$ is in $\mathbb{R}^{n*}$, then we get the hyperplane

$$H_u = \{v \in \mathbb{R}^n \mid \langle u, v\rangle = 0\}$$

and the closed half-space

$$H_u^+ = \{v \in \mathbb{R}^n \mid \langle u, v\rangle \geq 0\}.$$

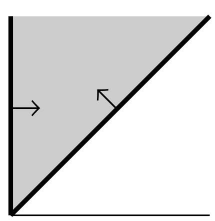
Exercise 1.4. Let $\sigma = \mathrm{Cone}(S)$ be a convex polyhedral cone in $\mathbb{R}^n$.

a. Prove that $\sigma^\vee = \{u \in \mathbb{R}^{n*} \mid \langle u, v\rangle \geq 0 \text{ for all } v \in S\}$.
b. Each $v \in \mathbb{R}^n$ gives a closed half-space $H_v^+ \subset \mathbb{R}^{n*}$. Prove that $\sigma^\vee = \bigcap_{v\in S} H_v^+$.

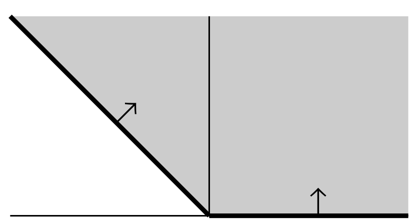
Here are some examples of dual cones.

Example 1.9. If $\sigma = \mathrm{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3)$ is the cone of Example 1.8, then part a of Exercise 1.4 implies that $\sigma^\vee$ is defined by the inequalities (1.1) from Example 1.5. □

Example 1.10. If $\sigma = \mathrm{Cone}(e_1 + e_2, e_2) \subset \mathbb{R}^2$, then σ and its dual can be pictured as follows:



a cone σ in the plane



the dual cone of σ

By part b of Exercise 1.4, the dual is $H_{e_1+e_2}^+ \cap H_{e_2}^+$. □

We next discuss the faces of a cone.

Definition 1.4. The set $\tau = H_u \cap \sigma$ is a ***face*** of the convex polyhedral cone σ if $u \in \mathbb{R}^{n*} \setminus \{0\}$ and $\sigma \subset H_u^+$. We regard σ as a face of itself. Faces $\tau \neq \sigma$ are called ***proper faces***.

Since $\sigma \subset H_u^+$ if and only if $u \in \sigma^\vee$, it follows that the faces of σ are given by $H_u \cap \sigma$ as u varies over the nonzero elements of the dual cone $\sigma^\vee$.

Lemma 1.5. *Let* $\sigma = \mathrm{Cone}(S)$ *be a convex polyhedral cone.*

(1) *Every face of* σ *is a convex polyhedral cone.*
(2) *An intersection of two faces of* σ *is again a face of* σ.
(3) *A face of a face of* σ *is again a face of* σ.

We say that τ is a *facet* of a convex polyhedral cone σ is τ is a face of codimension 1, i.e., $\dim \tau = \dim \sigma - 1$. One can prove that every proper face of σ is the intersection of the facets containing it. When we represent a facet as $\tau = H_u \cap \sigma$, we say that u is an *inward-pointing facet normal* or simply a *facet normal*.

Example 1.11. In Example 1.10, the generators of σ lie on the thick lines and the facet normals are represented by the two arrows. Note how the facet normals of σ become the generators of its dual, while the facet normals of the dual generate σ. □

This example generalizes as follows.

Theorem 1.6. *Suppose that $\sigma \subset \mathbb{R}^n$ is an n-dimensional convex polyhedral cone such that $\sigma \neq \mathbb{R}^n$. Let the facets of σ be $\tau_i = H_{u_i} \cap \sigma$, where $\sigma \subset H_{u_i}^+$ for $i = 1, \dots, s$. Then*

$$\sigma = H_{u_1}^+ \cap \cdots \cap H_{u_s}^+ \quad \textit{and} \quad \sigma^\vee = \operatorname{Cone}(u_1, \dots, u_s).$$

Furthermore, $(\sigma^\vee)^\vee = \sigma$.

1.5. **Strongly Convex Polyhedral Cones.** A convex polyhedral cone σ is *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$. There are several equivalent ways to think about strong convexity.

Proposition 1.7. *Let $\sigma \subset \mathbb{R}^n$ be a convex polyhedral cone. Then the following are equivalent:*
(1) *σ is strongly convex, i.e., $\sigma \cap (-\sigma) = \{0\}$.*
(2) *σ contains no positive-dimensional subspace.*
(3) *$\{0\}$ is a face of σ.*
(4) *$\dim \sigma^\vee = n$.*

One corollary of this proposition is that if σ is strongly convex of maximal dimension, then so is $\sigma^\vee$. The cones pictured in Examples 1.8 and 1.10 satisfy this condition.

Exercise 1.5. Suppose that $\sigma \subset \mathbb{R}^n$ is a convex polyhedral cone of dimension d, and assume that $W = \sigma \cap (-\sigma)$ has dimension $r > 0$. Let $\overline{\sigma} = (\sigma + W)/W \subset \mathbb{R}^n/W$.
a. Prove that $\overline{\sigma}$ is a strongly convex polyhedral cone in $\mathbb{R}^n/W$.
b. Prove that the map $\tau \mapsto (\tau + W)/W$ induces a one-to-one inclusion-preserving map from faces of σ to faces of $\overline{\sigma}$.

Exercise 1.6. Let $P \subset \mathbb{R}^n$ be a polytope and let $\sigma \subset \mathbb{R}^{n+1}$ be the convex polyhedral cone constructed in Exercise 1.2. Prove that σ is strictly convex.

An *edge* (or *ray*) of a convex polyhedral cone σ is a 1-dimensional face. When σ is strongly convex, we can use the edges to get an especially nice set of generators.

Proposition 1.8. *Let σ be a strongly convex polyhedral cone with edges $\rho_1, \dots, \rho_s$. Pick $v_i \in \rho_i \setminus \{0\}$ and set $S = \{v_1, \dots, v_s\}$. Then:*
(1) *$\sigma = \operatorname{Cone}(S)$.*
(2) *S is minimal in the sense that if T is any generating set for σ, then there are $\lambda_i > 0$ such that $\{\lambda_1 v_1, \dots, \lambda_s v_s\} \subset T$.*

In the situation of Proposition 1.8, we call S a *minimal generating set* of σ.

Exercise 1.7. Let σ be the cone over a polytope P as in Exercise 1.2. We define $v \in P$ to be a *vertex* if there is an affine half-space H^+ containing P such that $\{v\} = H \cap P$, where H is the boundary of H^+. Show that the vertices of P form a minimal generating set of σ.

Exercise 1.8. (This exercise is from [17].) Consider the cone $\sigma \subset \mathbb{R}^4$ generated by the standard basis e_1, e_2, e_3, e_4 together with the vectors $-e_i + 2\sum_{j \neq i} e_j$ for $1 \leq i \leq 4$.

a. Prove that σ is strongly convex, has dimension 4, and has 8 minimal generators.
b. Let e_i^* be the dual basis of $\mathbb{R}^{4*}$. Prove that the minimal generators of $\sigma^\vee$ are the 12 vectors $2e_i^* + e_j^*$ for $i \neq j$.

1.6. **Rational Polyhedral Cones.** Given a lattice N with dual M, we get dual vector spaces $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R} \simeq \mathbb{R}^n$ and $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R} \simeq \mathbb{R}^{n*}$. Everything we've said about cones in $\mathbb{R}^n$ carries over to cones in $N_\mathbb{R}$. But we now have one more ingredient, namely the lattice $N \subset N_\mathbb{R}$. This leads to the following definition.

Definition 1.9. $\sigma \subset N_\mathbb{R}$ is a ***rational polyhedral cone*** if $\sigma = \mathrm{Cone}(S)$ for a finite set $S \subset N$.

The cones pictured in Examples 1.8 and 1.10 are rational. Also, if σ is a rational polyhedral cone, then so are its faces and its dual.

The theory of toric varieties uses strongly convex rational polyhedral cones. These cones have uniquely determined minimal generated sets, described as follows. In Proposition 1.8, we saw that given such a cone σ, we get minimal generators by picking generators for each ray ρ of σ. Since ρ is a rational ray, it follows that the semigroup $\rho \cap N$ is generated by a unique element of the intersection, denoted v_ρ. The ray generators give the unique *minimal rational generating set* of a strongly convex rational polyhedral cone.

This implies a nice uniqueness result for convex rational polyhedral cones of maximal dimension. Namely, if $u_1, \ldots, u_s \in M$ are the minimal rational generators of $\sigma^\vee$, then

$$\sigma = H_{u_1}^+ \cap \cdots \cap H_{u_s}^+,$$

where the facet normals $u_i \in M$ are now uniquely determined.

The following definition will be important when studying smooth and simplicial toric varieties.

Definition 1.10. A strongly convex rational polyhedral cone $\sigma \subset N_\mathbb{R}$ is ***regular*** (resp. ***simplicial***) if its minimal generators form part of a $\mathbb{Z}$-basis of N (resp. $\mathbb{R}$-basis of $N_\mathbb{R}$).

In the literature, regular cones are sometimes called "smooth." The cone pictured in Example 1.8 is neither regular nor simplicial, though the cones in Example 1.10 are both regular.

1.7. **Affine Toric Varieties.** If $\sigma \subset N_\mathbb{R}$ is a rational polyhedral cone, then we define

$$S_\sigma = \sigma^\vee \cap M.$$

One easily sees that S_σ is a semigroup under addition with $0 \in S_\sigma$ as the additive identity. We also have Gordan's Lemma, which asserts the following.

Proposition 1.11. *If $\sigma \subset N_\mathbb{R}$ is a rational polyhedral cone, then $S_\sigma = \sigma^\vee \cap M$ is a finitely generated semigroup.*

We now associate to S_σ its *semigroup algebra* $\mathbb{C}[S_\sigma]$, which is defined as follows. As a $\mathbb{C}$-vector space, $\mathbb{C}[S_\sigma]$ has S_σ as a basis, with the basis vector corresponding to $m \in S_\sigma$ written symbolically as χ^m. Thus elements of $\mathbb{C}[S_\sigma]$ are formal linear combinations $\sum_{m \in S_\sigma} a_m \chi^m$, where only finitely many a_m are nonzero, and the product in $\mathbb{C}[S_\sigma]$ is determined by the "exponential rule"

$$\chi^m \cdot \chi^{m'} = \chi^{m+m'}$$

and the distributive law. Thus $\chi^0 = 1$ is the multiplicative unit of S_σ, and $\chi^m \in S_\sigma$ is invertible if and only if $-m \in S_\sigma$.

By Gordan's Lemma, S_σ has finitely many generators $m_1, \ldots, m_r$, which implies that $\mathbb{C}[S_\sigma]$ is generated by $\chi^{m_1}, \ldots, \chi^{m_r}$ as a $\mathbb{C}$-algebra. We also remark that if we pick a $\mathbb{Z}$-basis of N, then identifying χ^m with $t^m \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ gives an inclusion

$$\mathbb{C}[S_\sigma] \subset \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \tag{1.4}$$

of $\mathbb{C}$-algebras. It follows that $\mathbb{C}[S_\sigma]$ is an integral domain that is finitely generated as a $\mathbb{C}$-algebra. This allows us to make the following fundamental definition.

Definition 1.12. Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone. Then the irreducible affine variety

$$V_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$$

is the ***affine toric variety*** associated to σ.

In the following examples of affine toric varieties, we let $N = M = \mathbb{Z}^n$ with standard basis $e_1, \dots, e_n$.

Example 1.12. The cone $\sigma = \operatorname{Cone}(e_1, \dots, e_n) \subset \mathbb{R}^n$ is self-dual, so that $\sigma^\vee = \sigma$. Thus $\sigma^\vee \cap \mathbb{Z}^n = \mathbb{Z}^n_{\geq 0}$, and the resulting semigroup algebra is clearly $\mathbb{C}[x_1, \dots, x_n]$. It follows that $V_\sigma = \mathbb{C}^n$. □

Example 1.13. Fix an integer $0 < d < n$ and let $\sigma = \operatorname{Cone}(e_1, \dots, e_d) \subset \mathbb{R}^n$. One computes that $\sigma^\vee = \operatorname{Cone}(e_1, \dots, e_d, \pm e_{d_1}, \dots, \pm e_n)$, so that $\mathbb{C}[S_\sigma] = \mathbb{C}[x_1, \dots, x_d, x_{d+1}^{\pm 1}, \dots, x_n^{\pm 1}]$. Since $\operatorname{Spec}(\mathbb{C}[x]) = \mathbb{C}$ and $\operatorname{Spec}(\mathbb{C}[x, x^{-1}]) = \mathbb{C}^*$, it follows that $V_\sigma = \mathbb{C}^d \times (\mathbb{C}^*)^{n-d}$. □

Example 1.14. The cone $\sigma = \{0\}$ clearly has $\sigma^\vee = \mathbb{R}^n$, so that the resulting semigroup algebra is $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. It follows that $V_\sigma = (\mathbb{C}^*)^n$. □

Example 1.15. Let $\sigma = \operatorname{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3) \subset \mathbb{R}^3$ be the cone from Example 1.8. The inward pointing normals of the facets of σ are

$$m_1 = (1,0,0),\ m_2 = (0,1,0),\ m_3 = (0,0,1),\ m_4 = (1,1,-1),$$

which by Theorem 1.6 means that m_1, m_2, m_3, m_4 generate $\sigma^\vee$. In this case they also generate $S_\sigma = \sigma^\vee \cap \mathbb{Z}^3$. To describe $V_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$, we use the $\mathbb{C}$-algebra homomorphism $\mathbb{C}[x,y,z,w] \to \mathbb{C}[S_\sigma]$ defined by

$$x \mapsto t^{m_1},\ y \mapsto t^{m_2},\ z \mapsto t^{m_3},\ w \mapsto t^{m_4}.$$

Then $xy - zw \mapsto 0$ since $m_1 + m_2 = m_3 + m_4$, and one proves without difficulty that

$$\mathbb{C}[x,y,z,w]/\langle xy - zw\rangle \simeq \mathbb{C}[S_\sigma].$$

It follows that $V_\sigma \simeq \mathbf{V}(xy - zw) \subset \mathbb{C}^4$.

In Example 1.9, we noted that $\sigma^\vee$ is defined by the inequalities (1.1) from Example 1.5. Recall that in this example, we asked which Laurent monomials extend to functions on $\mathbf{V}(xy - zw)$. We now see that $S_\sigma = \sigma^\vee \cap \mathbb{Z}^3$ answers this question and determines the variety completely.

Exercise 1.9. In Exercise 1.1, you showed that $\mathbb{P}(1,1,2)$ is the union of affine toric varieties $U_0 \simeq \mathbb{C}^2$, $U_1 \simeq \mathbb{C}^2$, and $U_2 \simeq \mathbf{V}(xz - y^2) \subset \mathbb{C}^3$. Show that $\mathbf{V}(xz - y^2)$ is isomorphic to the affine toric variety V_σ for the cone $\sigma = \operatorname{Cone}(e_1, e_1 + 2e_2)$.

Our final task is to prove that V_σ is a toric variety in the sense of Definition 1.1.

Theorem 1.13. *Let $\sigma \subset N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a strongly convex rational polyhedral cone and let $V_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$, $S_\sigma = \sigma^\vee \cap M$. Then V_σ is a normal toric variety of dimension n.*

Proof. By Gordan's lemma, semigroup S_σ is finitely generated, say by $m_1, \dots, m_r$. Combining this with (1.4), we get $\mathbb{C}$-algebra homomorphisms

$$\mathbb{C}[x_1, \dots, x_r] \xrightarrow{\alpha} \mathbb{C}[S_\sigma] \xrightarrow{\beta} \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}],$$

where α is surjective and β uses an isomorphism $M \simeq \mathbb{Z}^n$ to identify the character χ^m with the Laurent monomial t^m. Letting I denote the kernel of α, we obtain maps of varieties

$$(\mathbb{C}^*)^n \xrightarrow{\beta^*} V_\sigma \xrightarrow{\alpha^*} \mathbf{V}(I) \subset \mathbb{C}^r.$$

To understand β^*, note that by Proposition 1.7, $\sigma^\vee$ has dimension n since σ is strongly convex. Hence we can pick m_0 in the interior of $\sigma^\vee$. Given any $m \in M$, one easily sees that $m + \ell m_0 \in \sigma^\vee$ for $\ell \gg 0$. Hence $t^m = \frac{t^{m+\ell m_0}}{(t^{m_0})^\ell} \in \mathbb{C}[S_\sigma]_{t^{m_0}}$, where $\mathbb{C}[S_\sigma]_{t^{m_0}}$ is the localization defined in Section 0.4 of the BACKGROUND. Thus β factors

$$\mathbb{C}[S_\sigma] \subset \mathbb{C}[S_\sigma]_{t^{m_0}} = \mathbb{C}[M] \simeq \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}],$$

so that we can regard $(\mathbb{C}^*)^n$ as a Zariski open subset of V_σ via β^*. It follows that $\dim V_\sigma = n$.

The above map $(\mathbb{C}^*)^n \to \mathbb{C}^r$ is given by

$$(t_1, \dots, t_n) \longmapsto (t^{m_1}, \dots, t^{m_r}). \tag{1.5}$$

This is injective and the action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^r$ defined by

$$(t_1, \dots, t_n) \cdot (x_1, \dots, x_r) = (t^{m_1} x_1, \dots, t^{m_r} x_r).$$

Since $(\mathbb{C}^*)^n$ is Zariski dense in V_σ, it follows that $\mathbf{V}(I)$ is the Zariski closure of the image of (1.5). This in turn implies that V_σ is stable under the above $(\mathbb{C}^*)^n$-action. This proves that V_σ satisfies the conditions of Definition 1.1 and hence is a toric variety.

It remains to prove that V_σ is normal, i.e., that $\mathbb{C}[S_\sigma]$ is integrally closed. Recall that $\sigma^\vee = \bigcap_{i=1}^s H_{v_i}^+$, where $v_1, \dots, v_s$ are the minimal generators of σ. If we set $\tau_i = \mathrm{Cone}(v_i)$, then

$$\mathbb{C}[S_\sigma] = \mathbb{C}[\sigma^\vee \cap M] = \bigcap_{i=1}^s \mathbb{C}[\tau_i^\vee \cap M] = \bigcap_{i=1}^s \mathbb{C}[S_{\tau_i}].$$

Since v_i can be taken to be the first element of a $\mathbb{Z}$-basis of N, Example 1.13 shows that

$$\mathbb{C}[S_{\tau_i}] \simeq \mathbb{C}[x_1, x_2^{\pm 1}, \dots, x_n^{\pm 1}].$$

The ring on the right-hand side is easily seen to be a UFD and hence is normal. Then the same is true for $\mathbb{C}[S_{\tau_i}]$. It follows that $\mathbb{C}[S_\sigma]$ is normal since an intersection of normal domains with the same field of fractions is normal. □

The above proof gives a concrete way to think about the affine toric variety V_σ. Namely, generators $m_1, \dots, m_r$ of $S_\sigma = \sigma^\vee \cap M$ give the monomial embedding

$$(\mathbb{C}^*)^n \longrightarrow \mathbb{C}^r$$

from (1.5). Then V_σ is the Zariski closure of the image of this map. The proof also describes how $(\mathbb{C}^*)^n$ acts on V_σ. We will generalize this approach in LECTURE 3 when we construct (possibly non-normal) affine and projective toric varieties using the Zariski closure of monomial maps. A more abstract way to show that V_σ is a toric variety is given in Section 1.3 of [17].

Theorem 1.13 describes *all* normal affine varieties that are toric. Here is the precise result.

Theorem 1.14. *Let V be an affine variety that is also a toric variety. Then V is isomorphic to $V_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ for some strongly convex rational polyhedral cone σ if and only if V is normal.*

Here is an exercise to illustrate the role of normality.

Exercise 1.10. The toric variety $C = \mathbf{V}(y^2 - x^3)$ from Example 1.4 contains $\mathbb{C}^*$ via $t \mapsto (t^2, t^3)$.

a. Show that C is isomorphic to the affine variety $\mathrm{Spec}(\mathbb{C}[S])$, where $\mathbb{C}[S]$ is the semigroup algebra of the semigroup $S = \{0, 2, 3, \dots\} \subset \mathbb{Z}$.
b. In general, a subsemigroup $S \subset M$ is *saturated* if $km \in S$ implies $m \in S$ for all $m \in M$ and $k > 0$ in $\mathbb{Z}$. Prove that $\sigma^\vee \cap M$ is saturated when σ is a rational polyhedral cone in $N_\mathbb{R}$.
c. Prove that the semigroup S of part a is not saturated.

Exercise 1.11. Let $S \subset M$ be a finitely generated subsemigroup. Prove that $\mathbb{C}[S]$ is normal $\Leftrightarrow$ S is saturated $\Leftrightarrow$ $S = \sigma^\vee \cap M$ for some rational polyhedral cone $\sigma \subset N_\mathbb{R}$.

Lecture 2. The Toric Variety of a Fan

We define the toric variety of a fan and then discuss the basic properties of toric varieties.

2.1. **Fans and Their Toric Varieties.** In Lecture 1, we defined the affine toric variety V_σ of a strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}} \simeq \mathbb{R}^n$. Our next step is to create more general toric varieties by gluing together affine toric varieties containing the same torus $T(N) \simeq (\mathbb{C}^*)^n$. This brings us to the concept of a *fan*, which is defined to be a finite collection Σ of cones in $N_{\mathbb{R}}$ with the following three properties:

(1) Each $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.
(2) If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$.
(3) If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of each.

The basic idea is that a fan Σ encodes the information needed to glue together the affine toric varieties V_σ, $\sigma \in \Sigma$ to create an abstract variety X_Σ. To make this work, we need to study how affine toric varieties fit together.

Given a fan Σ, let $\sigma \in \Sigma$ and let τ be a face of σ. Then $\tau \subset \sigma$ induces $\mathbb{C}[S_\sigma] \subset \mathbb{C}[S_\tau]$, which gives a map $V_\tau \to V_\sigma$. To understand this map, note that $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee \cap M = S_\sigma$. Then one can prove that $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-m)$, which implies that $\mathbb{C}[S_\sigma] \subset \mathbb{C}[S_\tau]$ can be written

$$\mathbb{C}[S_\sigma] \subset \mathbb{C}[S_\sigma]_{\chi^m} = \mathbb{C}[S_\tau].$$

Thus V_τ is naturally isomorphic to the Zariski open subset of V_σ defined by $\chi^m \neq 0$.

Cones $\sigma, \sigma' \in \Sigma$ have the common face $\sigma \cap \sigma'$. In this case, we get open immersions

$$V_{\sigma\cap\sigma'} \longrightarrow V_\sigma$$
$$V_{\sigma\cap\sigma'} \longrightarrow V_{\sigma'}.$$

The images of these maps will be denoted $V_{\sigma\sigma'}$ and $V_{\sigma'\sigma}$ respectively. Then we have an isomorphism

$$g_{\sigma\sigma'} : V_{\sigma\sigma'} \simeq V_{\sigma'\sigma}.$$

This gives gluing data $\{V_\sigma, V_{\sigma\sigma'}, g_{\sigma\sigma'}\}$ as described in Section 0.12 of the Background.

Definition 2.1. Given a fan Σ in $N_{\mathbb{R}}$, X_Σ is the abstract variety constructed using the above gluing data.

In more down-to-earth terms, X_Σ is constructed from the affine varieties V_σ, $\sigma \in \Sigma$, by gluing V_σ and $V_{\sigma'}$ along their common open subset $V_{\sigma\cap\sigma'}$ for all $\sigma, \sigma' \in \Sigma$.

Theorem 2.2. *The variety X_Σ is a separated normal toric variety.*

Proof. Each $\sigma \in \Sigma$ gives an affine open subset $V_\sigma \subset X_\Sigma$. In particular, the trivial cone $\{0\} \in \Sigma$ gives an open subset $T(N) \subset X_\Sigma$. Furthermore, since $\{0\}$ is a face of every $\sigma \in \Sigma$, we get compatible inclusions $T(N) \subset V_\sigma$. Then:

- Since $T(N)$ is Zariski dense in each V_σ, it is Zariski dense in X_Σ. Thus X_Σ is irreducible.
- Since $T(N)$ acts on each V_σ and the gluing data is $T(N)$-equivariant, $T(N)$ acts on X_Σ.

This proves that X_Σ satisfies Definition 1.1 from Lecture 1.

It is easy to see that X_Σ is normal since normality is a local property and each V_σ is normal. Finally, we need to show that X_Σ is separated, i.e., that the diagonal map $X_\Sigma \to X_\Sigma \times X_\Sigma$ has closed image. By looking at the gluing data, this reduces to checking that the diagonal map $V_{\sigma\cap\sigma'} \to V_\sigma \times V_{\sigma'}$ is a closed embedding, which is true provided that the natural map

$$\mathbb{C}[S_\sigma] \otimes_{\mathbb{C}} \mathbb{C}[S_{\sigma'}] \to \mathbb{C}[S_{\sigma\cap\sigma'}]$$

is surjective. This follows from the equality

$$S_{\sigma\cap\sigma'} = S_\sigma + S_{\sigma'}$$

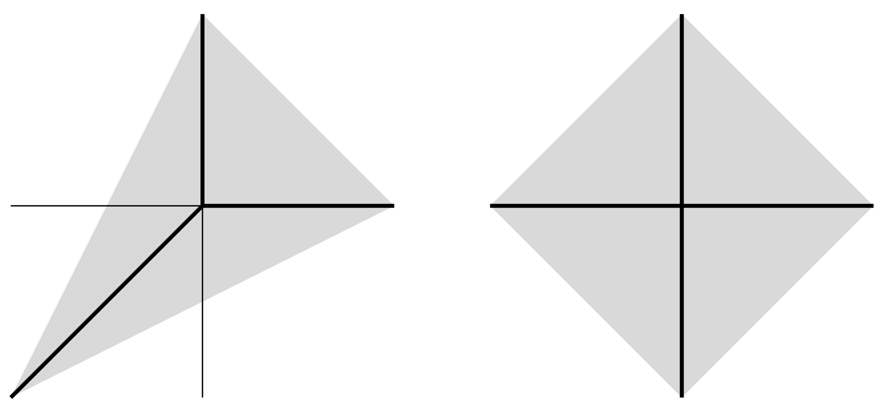
proved in Section 1.2 of [17]. □

In [23, §I.2, Thm. 5], it is shown that *all* normal toric varieties arise in this way, i.e., every normal toric variety is determined by a fan. These toric varieties are sometimes called *torus embeddings*, and [17] and [26] call the fan Δ. Also, the toric variety determined by Σ is variously denoted X_Σ, $X(\Sigma)$, $Z(\Sigma)$, and $T_N\mathrm{emb}(\Sigma)$. Furthermore, polytopes (which we will encounter in LECTURE 4) are denoted P, $\square$, and (just to confuse matters more) Δ. The lack of uniform notation is unfortunate, so that the reader of a paper using toric methods needs to look carefully at the notation.

2.2. **Examples.** Here are some examples of toric varieties.

Example 2.1. Given $\sigma \subset N_{\mathbb{R}}$, we get a fan by taking all faces of σ (including σ). One easily sees that the toric variety of this fan is the affine toric variety V_σ. □

Example 2.2. The fans for $\mathbb{P}^2$ (on the left) and $\mathbb{P}^1 \times \mathbb{P}^1$ (on the right) are as follows:



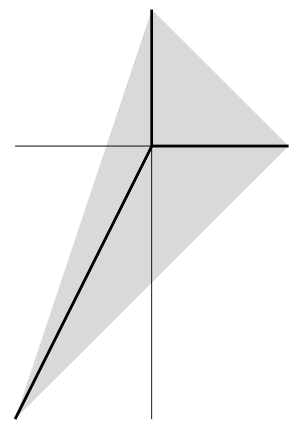
Here, the 1-dimensional cones are indicated with thick lines, and 2-dimensional cones (which extend to infinity) are shaded. Thus the fan for $\mathbb{P}^2$ has three 2-dimensional cones, while the fan for $\mathbb{P}^1 \times \mathbb{P}^1$ has four such cones.

To see that the fan on the left gives $\mathbb{P}^2$, observe that the 2-dimensional cones $\sigma_1, \sigma_2, \sigma_3$ are generated by bases of $\mathbb{Z}^2$. As noted in Example 1.12 from LECTURE 1, this implies that the affine toric varieties V_{σ_i} are copies of $\mathbb{C}^2$. By checking how these fit together along $V_{\sigma_i\cap\sigma_j}$, one gets the usual way of constructing $\mathbb{P}^2$ by gluing together three copies of $\mathbb{C}^2$. Note that this fan appeared earlier in (1.2) of Example 1.7 from LECTURE 1.

A similar argument shows that the fan on the right gives $\mathbb{P}^1 \times \mathbb{P}^1$. □

Example 2.3. Let $e_1, \ldots, e_n$ be a basis of $N = \mathbb{Z}^n$, and set $e_0 = -e_1 - \cdots - e_n$. Then we get a fan by taking the cones generated by all proper subsets of $\{e_0, e_1, \ldots, e_n\}$. We leave it as an exercise for the reader to show that the associated toric variety is $\mathbb{P}^n$. When $n = 2$, this gives the fan on the left in Example 2.2. □

Example 2.4. Let e_1, e_2 be a basis of $N = \mathbb{Z}^2$, and set $e_0 = -e_1 - 2e_2$. This gives the fan Σ:



We claim that the corresponding toric variety X_Σ is the weighted projective space $\mathbb{P}(1,1,2)$. To see this, let $\sigma_0 = \mathrm{Cone}(e_1, e_2)$, $\sigma_1 = \mathrm{Cone}(e_0, e_2)$, and $\sigma_2 = \mathrm{Cone}(e_0, e_1)$. Then X_Σ is the union of the affine toric varieties V_{σ_i}, $i = 1, 2, 3$. Then $V_{\sigma_0} \simeq V_{\sigma_1} \simeq \mathbb{C}^2$ since σ_1 and σ_2 are smooth. However, $\sigma_2 = \mathrm{Cone}(-e_1 - e_2, e_1)$, which is $\mathrm{GL}_2(\mathbb{Z})$-equivalent to the cone $\mathrm{Cone}(e_1, e_1 + 2e_2)$ of Exercise 1.9 from LECTURE 1. Hence the V_{σ_i} are isomorphic to the affine open subsets U_i of Example 1.1 from LECTURE 1. Since they glue the same way, we obtain $X_\Sigma \simeq \mathbb{P}(1,1,2)$. □

Example 2.5. Let $q_0, \ldots, q_n$ be positive integers with $\gcd(q_0, \ldots, q_n) = 1$. The fan for the weighted projective space $\mathbb{P}(q_0, \ldots, q_n)$ is described as follows. Let $N = \mathbb{Z}^{n+1}/\mathbb{Z}(q_0, \ldots, q_n)$ and let $v_i \in N$ be the image of the standard basis vector $e_i \in \mathbb{Z}^{n+1}$ for $i = 0, \ldots, n$. Then $q_0 v_0 + \cdots + q_n v_n = 0$, and we get a fan in $N_\mathbb{R}$ by taking the cones generated by all proper subsets of $\{v_0, v_1, \ldots, v_n\}$.

When $q_i = 1$ for all i, the vectors $v_1, \ldots, v_n$ form a $\mathbb{Z}$-basis of N and $v_0 = -v_1 - \cdots - v_n$. Thus we recover the fan of Example 2.3, which gives $\mathbb{P}^n = \mathbb{P}(1, \ldots, 1)$. On the other hand, when $n = 2$ with $q_0 = q_1 = 1$ and $q_2 = 2$, the vectors v_1, v_2 form a $\mathbb{Z}$-basis of N and $v_0 = -v_1 - 2v_2$. Thus we recover the fan of Example 2.4, which gives $\mathbb{P}(1,1,2)$. For general $q_0, \ldots, q_n$, one can show that the above fan gives $\mathbb{P}(q_0, \ldots, q_n)$. □

Exercise 2.1. Let Σ_1 and Σ_2 be fans in $(N_1)_\mathbb{R}$ and $(N_2)_\mathbb{R}$ respectively, and let $N = N_1 \oplus N_2$. Prove that the set of cones

$$\Sigma = \{\sigma_1 \times \sigma_2 \mid \sigma_i \in \Sigma_i,\ i = 1, 2\}$$

is a fan in $N_\mathbb{R}$ and that X_Σ is naturally isomorphic to the cartesian product $X_{\Sigma_1} \times X_{\Sigma_2}$. Also explain how the fan on right in Example 2.2 relates to this construction.

There are *many* other nice examples of toric varieties, including Hirzebruch surfaces, rational normal scrolls, and equivariant projective bundles over projective spaces. We will see in LECTURE 4 that every lattice polytope in $M_\mathbb{R}$ determines a projective toric variety.

2.3. **Orbits and Faces.** We next consider the combinatorial structure of normal toric varieties. The basic idea is that we can generalize the relation between cones and limits discussed in Example 1.7. More precisely, there are one-to-one correspondences between the following sets of objects:

- The limits $\lim_{t \to 0} \lambda^u(t)$ for $u \in |\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$ ($|\Sigma|$ is the *support* of Σ).
- The cones $\sigma \in \Sigma$.
- The orbits O of T on X_Σ.

The correspondences work as follows: an orbit O corresponds to a cone σ if and only if $\lim_{t\to 0}\lambda^u(t)$ exists and lies in O for all u in the relative interior of σ. If $\mathrm{orb}(\sigma)$ is the orbit corresponding to σ, then one has

- $\dim\sigma + \dim\mathrm{orb}(\sigma) = n$.
- $\mathrm{orb}(\sigma) \subset \overline{\mathrm{orb}(\tau)}$ if and only if $\tau \subset \sigma$.

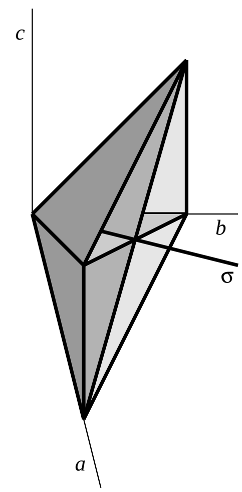
In particular, fixed points of the torus action correspond to n-dimensional cones in the fan.

If we work with the orbit closures $\overline{\mathrm{orb}(\sigma)}$, we get similar correspondences. In addition, each orbit closure is a toric variety, and its fan can be described as follows. Given σ, let $N_\sigma = \mathbb{R}\sigma \cap N$, where $\mathbb{R}\sigma$ is the span of σ (see Section 1.4 of Lecture 1). Then N/N_σ is naturally dual to $\sigma^\perp \cap M$. Further, any cone $\tau \in \Sigma$ containing σ gives the cone

$$\overline{\tau} = (\tau + \mathbb{R}\sigma)/\mathbb{R}\sigma = (\tau + (N_\sigma)_\mathbb{R})/(N_\sigma)_\mathbb{R} \subset (N/N_\sigma)_\mathbb{R}.$$

These cones form a fan in $(N/N_\sigma)_\mathbb{R}$ which gives the toric variety $\overline{\mathrm{orb}(\sigma)}$. Geometrically, this fan is obtained from the *star* of σ (= all cones of Σ containing σ) by collapsing σ to a point in order to create a new fan in $(N/N_\sigma)_\mathbb{R}$.

Example 2.6. Subdividing the cone of Example 1.8 from Lecture 1 gives the following fan:



The corresponding toric variety X has dimension 3, so that the orbit corresponding to the ray σ pictured above has dimension 2. The star of σ is the whole fan, and when we collapse σ to a point, the new fan we get is clearly the fan on the right in Example 2.2. It follows that $\overline{\mathrm{orb}(\sigma)}$ is $\mathbb{P}^1 \times \mathbb{P}^1$.

We remark that the affine toric variety of the original cone (before subdividing) is singular. The subdivided cone is a resolution of singularities of the affine toric variety. This, however, is not the simplest way to resolve the singularity. See [6] for other resolutions and a proof of resolution of singularities for arbitrary toric varieties. □

2.4. **Properties of Toric Varieties.** Here are some basic properties of normal toric varieties. Proofs can be found in [26, Thm. 1.11, Thm. 1.10 and Cor. 3.9].

Theorem 2.3. *Let X_Σ be the toric variety determined by a fan Σ in $N_\mathbb{R}$. Then:*

(1) *X_Σ is complete $\iff$ Σ is complete, i.e., $|\Sigma| = N_\mathbb{R}$.*
(2) *X_Σ is smooth $\iff$ every $\sigma \in \Sigma$ is a smooth cone, i.e., is generated by a subset of a $\mathbb{Z}$-basis of N.*
(3) *X_Σ is Cohen-Macaulay.*
(4) *X_Σ has at worst rational singularities.*

Remark 2.4. Here are some comments on Theorem 2.3.

(1) The term "complete" applies varieties over any field (see [15, 21]), though over $\mathbb{C}$, "complete" is equivalent to being compact in the classical topology. Thus projective varieties are complete, though the converse can fail. The books [17, p. 71] and [26, pp. 85–86] describe a classic example of a smooth toric variety that is complete but not projective. In LECTURE 4 we will give a criteron for X_Σ to be projective.
(2) Example 1.13 shows that X_Σ is smooth when every cone in Σ is smooth.
(3) The Cohen-Macaulay property of a toric variety X_Σ is useful, for it means that Grothendieck duality is especially nice for X_Σ. See [26, Section 3.2] for a careful explanation, and for a discussion of how Cohen-Macaulay relates to depth, see [13, Ch. 18]
(4) Rational singularities are a well-known class of singularities characterized by the vanishing of certain cohomology groups.

2.5. **Finite Quotient Singularities.** Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. Then G acts on both $\mathbb{C}^n$ and $\mathbb{C}[x_1,\ldots,x_n]$, and the quotient $\mathbb{C}^n/G$ is the set of G-orbits. By Chapter 7 of [9], we can turn this set into an affine variety as follows.

Proposition 2.5. *Given a finite subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$, let $\mathbb{C}[x_1,\ldots,x_n]^G \subset \mathbb{C}[x_1,\ldots,x_n]$ be the subring of invariant polynomials. Then there is a natural bijection $\mathbb{C}^n/G \simeq \mathrm{Spec}(\mathbb{C}[x_1,\ldots,x_n]^G)$.*

Understanding the structure of $\mathbb{C}[x_1,\ldots,x_n]^G$ is one of the goals of *invariant theory*.

Definition 2.6. A point p of a variety X is a ***finite quotient singularity*** if there is a finite subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$ such that $p \in X$ is analytically equivalent to $0 \in \mathbb{C}^n/G$. Then X is ***quasi-smooth*** or has ***finite quotient singularities*** if every point of p is a finite quotient singularity.

(By *analytically equivalent*, we mean a bijection between classical neighborhoods of the two points such that the map and its inverse are represented by convergent power series.)

Note that the definition of finite quotient singularity allows G to be the trivial subgroup of $\mathrm{GL}_n(\mathbb{C})$. It follows that any smooth variety is quasismooth. Here is an example to show that the converse is not true.

Example 2.7. Let $G = \{\pm I\} \subset \mathrm{GL}_2(\mathbb{C})$. If we think of $\mathbb{C}^2$ as $\mathrm{Spec}(\mathbb{C}[t_1,t_2])$, then $\mathbb{C}[t_1,t_2]^G = \mathbb{C}[t_1^2, t_1t_2, t_2^2]$, which is the semigroup algebra of the cone $\sigma = \mathrm{Cone}(e_1, e_1 + 2e_2)$. You showed in Exercise 1.9 that the corresponding affine toric variety is $V = \mathbf{V}(xz - y^2) \subset \mathbb{C}^3$. Thus $\mathbb{C}^2/G \simeq V$. It is easy to check that the origin is the unique singular point of V. Thus V is quasismooth but not smooth.

We also have the following basic result.

Proposition 2.7. *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite subgroup. Then $\mathbb{C}^n/G$ is quasismooth.*

The definition of quasismooth guarantees that $0 \in \mathbb{C}^n/G$ is quasismooth, but one still needs to show that the other points of $\mathbb{C}^n/G$ are quasismooth.

In some cases, the quotient $\mathbb{C}^n/G$ is still smooth, even when G is nontrivial.

Example 2.8. Let $C_m \in \mathrm{GL}_n(\mathbb{C})$ be the matrix with $e^{2\pi i/m}, 1, \ldots, 1$ on the main diagonal and 0's elsewhere, and let $G = \{C_m^i \mid 0 \le i \le m-1\}$. Since $\mathbb{C}[x_1,\ldots,x_n]^G = \mathbb{C}[x_1^m, x_2, \ldots, x_n]$, we see that $\mathbb{C}^n/G \simeq \mathbb{C}^n$.

A matrix in $\mathrm{GL}_n(\mathbb{C})$ is a *complex reflection* if it is conjugate to the matrix C_m of Example 2.8, and $G \subset \mathrm{GL}_n(\mathbb{C})$ is a *complex reflection group* if it is generated by complex reflections. The Shephard-Todd-Chevalley theorem says that $\mathbb{C}^n/G \simeq \mathbb{C}^n$ if and only if G is a complex reflection group.

A finite subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$ is *small* if it contains no complex reflections other than the identity. One can prove that if $G \subset \mathrm{GL}_n(\mathbb{C})$, then its complex reflections generate a normal subgroup H such that G/H is isomorphic to a small subgroup of $\mathrm{GL}_n(\mathbb{C})$. Since $\mathbb{C}^n/G \simeq (\mathbb{C}^n/H)/(G/H)$ and $\mathbb{C}^n/H \simeq \mathbb{C}^n$ by the Shephard-Todd-Chevalley theorem, one can always reduce to the case of a quotient by a small subgroup.

2.6. **Simplicial Toric Varieties.** Recall that a rational polyhedral cone is *simplicial* if its minimal generators are linearly independent over $\mathbb{R}$. Then a toric variety X_Σ is *simplicial* if every cone in Σ is simplicial.

The main result concerning simplicial toric varieties is as follows.

Theorem 2.8. *Let X_Σ be a toric variety. Then the following are equivalent:*

(1) *X_Σ is simplicial.*
(2) *X_Σ has finite Abelian quotient singularities, i.e., its singularities are analytically equivalent to $0 \in \mathbb{C}^n/G$ where $G \subset \mathrm{GL}_n(\mathbb{C})$ is an Abelian small subgroup.*
(3) *X_Σ has finite quotient singularities, i.e., X_Σ is quasismooth.*

We omit the proof, though it is useful to say a few words about (1) $\Rightarrow$ (2). Let $\sigma \subset N_\mathbb{R} \simeq \mathbb{R}^n$ be an n-dimensional simplicial cone. Then its minimal generators $v_1, \dots, v_n$ generate a sublattice $N' \subset N$ of finite index. Let $G = N/N'$ denote the quotient, which is a finite Abelian group. We claim that there is an action of G on $\mathbb{C}^n$ such that

$$V_\sigma \simeq \mathbb{C}^n/G.$$

To see why, let $M' = \mathrm{Hom}_\mathbb{Z}(N', \mathbb{Z})$, and note that $N' \subset N$ induces $M \subset M'$. Note that $\sigma \subset (N')_\mathbb{R}$ is a smooth cone, which implies that $\mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M']) \simeq \mathbb{C}^n$.

We also have an action of $G = N/N'$ on $\mathbb{C}[\sigma^\vee \cap M']$ defined by

$$(v + N') \cdot \chi^{u'} = e^{2\pi i \langle u', v\rangle}\, \chi^{u'}.$$

This induces an action of G on $\mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M']) \simeq \mathbb{C}^n$.

The key point is to prove that $\mathbb{C}[\sigma^\vee \cap M']^G = \mathbb{C}[\sigma^\vee \cap M]$. Once we have this, taking Spec gives

$$V_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M]) = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M']^G) = \mathbb{C}^n/G,$$

where the last equality is by Proposition 2.5, and our claim follows.

Example 2.9. Let Σ be a complete fan in $N_\mathbb{R} \simeq \mathbb{R}^2$. Then X_Σ is a complete surface (in fact, as we will see in Lecture 4, it is projective). Since 2-dimensional cones are simplicial, we see that X_Σ is quasismooth. Furthermore, each 1-dimensional cone is smooth since it is generated by a *primitive* element of N (meaning that it is not of the form ℓu for u in N and $\ell > 1$ in $\mathbb{Z}$). Hence the only singular points of X_Σ are the fixed points of the torus action corresponding to those 2-dimesional cones of Σ which are not smooth.

2.7. **Changing the Lattice.** Suppose that N is a lattice and $N' \subset N$ is a lattice of finite index. This implies that $N'_\mathbb{R} = N_\mathbb{R}$, and one also sees that a cone σ is strongly convex rational polyhedral in $N_\mathbb{R}$ if and only if it has the same property in $N'_\mathbb{R}$. When σ satisfies these conditions, we get affine toric varieties

$$V_{\sigma,N} = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$$
$$V_{\sigma,N'} = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M'])$$

where as above $M' = \mathrm{Hom}_\mathbb{Z}(N', \mathbb{Z})$ and $M \subset M'$. The latter induces an inclusion $\mathbb{C}[\sigma^\vee \cap M] \subset \mathbb{C}[\sigma^\vee \cap M']$, which in turn gives a map

$$V_{\sigma,N'} \to V_{\sigma,N}.$$

On level of tori, this restricts to the map $T(N') \to T(N)$ coming from $N' \subset N$. Since $\mathbb{C}$ is divisible and N/N' is finite, it is easy to see that $T(N') \to T(N)$ is surjective with kernel isomorphic to N/N'. Hence the action of $T(N')$ on $V_{\sigma,N'}$ induces an action of $G = N/N'$ on $V_{\sigma,N'}$. Generalizing the argument given above, one finds that

$$V_{\sigma,N'}/G \simeq V_{\sigma,N}. \tag{2.1}$$

Even more generally, if Σ is a fan in $N_{\mathbb{R}}$, then it is also a fan in $N'_{\mathbb{R}}$, and $G = N/N'$ acts on $X_{\Sigma,N'}$ with quotient

$$X_{\Sigma,N'}/G \simeq X_{\Sigma,N}. \tag{2.2}$$

Example 2.10. In Example 2.5, we constructed a fan Σ giving $\mathbb{P}(q_0, \ldots, q_n)$ using vectors $v_0, \ldots, v_n$ in the lattice $N = \mathbb{Z}^{n+1}/\mathbb{Z}(q_0, \ldots, q_n)$. Let $N' \subset N$ be the sublattice generated by $w_i = q_i v_i$ for $0 \le i \le n$. Then $\sum_{i=0}^n q_i v_i = 0$ implies that $\sum_{i=0}^n w_i = 0$, so that $w_0 = -w_1 - \cdots - w_n$, where $w_1, \ldots, w_n$ form a $\mathbb{Z}$-basis of N'. It follows that $X_{\Sigma,N'} = \mathbb{P}^n$ while $X_{\Sigma,N} = \mathbb{P}(q_0, \ldots, q_n)$. Hence the quotient map (2.2) shows that if we set $G = N/N'$, then

$$\mathbb{P}^n/G \simeq \mathbb{P}(q_0, \ldots, q_n).$$

You should check how this generalizes Exercise 0.2 from the BACKGROUND. □

LECTURE 3. HOMOGENEOUS COORDINATES AND TORIC IDEALS

In addition to the classic method of defining toric varieties by gluing together affine toric varieties (as done in LECTURE 2), two other techniques for constructing toric varieties have been given in recent years. These methods are:

- Generalized homogeneous coordinates.
- Toric ideals.

3.1. **Weil Divisors on Toric Varieties.**

We saw in Section 2.3 of LECTURE 2 that the rays (i.e., 1-dimensional cones) of a fan Σ in $N_{\mathbb{R}}$ correspond to the codimension 1 orbit closures in the normal toric variety X_Σ. If $\Sigma(1)$ is the set of all rays of Σ, then we will let D_ρ denote the irreducible torus-invariant divisor corresponding to $\rho \in \Sigma(1)$. Note that the torus of X_Σ is $T(N) = X_\Sigma \setminus \bigcup_\rho D_\rho$ (in general, we use $\bigcup_\rho$, $\sum_\rho$, etc. to denote union, summation, etc. over all $\rho \in \Sigma(1)$).

There is a nice relation between the divisors D_ρ and the characters χ^m coming from $m \in M$. Since χ^m maps T to $\mathbb{C}^*$, we can regard χ^m as a rational function on X_Σ which is nonvanishing on T. Hence the divisor of χ^m is supported on $\bigcup_\rho D_\rho$. Since X_Σ is normal and D_ρ is irreducible, the order of vanishing ord_{D_ρ} is defined (see Section 0.14 in the BACKGROUND). By [17, Sect. 3.3], we have the wonderful formula

$$\mathrm{ord}_{D_\rho}(\chi^m) = \langle m, v_\rho\rangle.$$

where, as in Section 1.6 from LECTURE 1, v_ρ is the unique generator of $\rho \cap N$ for $\rho \in \Sigma(1)$. It follows that the divisor of χ^m is given by

$$\mathrm{div}(\chi^m) = \textstyle\sum_\rho \langle m, v_\rho\rangle D_\rho. \tag{3.1}$$

Another useful fact proved in [17, Sect. 3.4] is that the torus-invariant divisors generate the Chow group $A_{n-1}(X_\Sigma)$ of Weil divisors modulo linear equivalence (see Section 0.15 in the BACKGROUND). In fact, we get an exact sequence

$$M \xrightarrow{\alpha} \textstyle\bigoplus_\rho \mathbb{Z} D_\rho \xrightarrow{\beta} A_{n-1}(X_\Sigma) \longrightarrow 0, \tag{3.2}$$

where α is defined by (3.1) and β is the map taking a Weil divisor to its divisor class in the Chow group. Furthermore, if the rays of $\Sigma(1)$ span $N_{\mathbb{R}}$, then α is injective, so that (3.2) becomes a short exact sequence in this case.

Example 3.1. Consider $\mathbb{P}^1 \times \mathbb{P}^1$. Using the fan on the right in Example 2.2 from LECTURE 2, we see that the n_ρ's are $v_1 = e_1, v_2 = -e_1, v_3 = e_2, v_4 = -e_2$. If the corresponding divisors are D_1, D_2, D_3, D_4, then the exact sequence (3.2) becomes

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\alpha} \textstyle\bigoplus_{i=1}^4 \mathbb{Z} D_i \xrightarrow{\beta} \mathbb{Z}^2 \longrightarrow 0,$$

where

$$\begin{aligned} \alpha(a,b) &= aD_1 - aD_2 + bD_3 - bD_4, \\ \beta(a_1D_1 + \cdots + a_4D_4) &= (a_1 + a_2, a_3 + a_4). \end{aligned} \tag{3.3}$$

We will return to this example several times during the lecture. □

Example 3.2. For the weighted projective space $\mathbb{P}(q_0, \ldots, q_n)$ of Example 2.5, the lattices are $N = \mathbb{Z}^{n+1}/\mathbb{Z}(q_0, \ldots, q_n)$ and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. It follows that $M = \mathbb{Z}(q_0, \ldots, q_n)^\perp$, which gives the exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha} \mathbb{Z}^{n+1} \xrightarrow{\beta} \mathbb{Z} \longrightarrow 0,$$

where α is the inclusion $\mathbb{Z}(q_0,\ldots,q_n)^\perp \subset \mathbb{Z}^{n+1}$ and β is dot product with $(q_0,\ldots,q_n)$. If we replace $\mathbb{Z}^{n+1}$ with $\bigoplus_{i=0}^n \mathbb{Z}D_i$, then we obtain (3.2) for weighted projective space. □

3.2. **The Homogeneous Coordinate Ring.** We begin with the first construction on our list, which concerns homogeneous coordinates for toric varieties. Returning to our basic example of $\mathbb{P}^n$, the usual homogeneous coordinates give not only the graded ring $\mathbb{C}[x_0,\ldots,x_n]$ but also the quotient construction $\mathbb{P}^n \simeq (\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{C}^*$. Given an arbitrary toric variety X_Σ, we can generalize this as follows. For each $\rho \in \Sigma(1)$, introduce a variable x_ρ, which gives the polynomial ring

$$S = \mathbb{C}\big[x_\rho : \rho \in \Sigma(1)\big].$$

To grade this ring, note that a monomial $\Pi_\rho x_\rho^{a_\rho}$ gives a divisor $D = \sum_\rho a_\rho D_\rho$. If we write the monomial as x^D, its degree is defined to be $\deg(x^D) = \beta(D) \in A_{n-1}(X_\Sigma)$, where β is the map from (3.2) which takes a divisor to its class in the Chow group. The ring S with this grading is called the *homogeneous coordinate ring* of X_Σ. Note that when X_Σ is smooth, the grading is by $\mathrm{Pic}(X_\Sigma)$ (see Section 0.5 from the BACKGROUND).

Example 3.3. We studied $\mathbb{P}^1 \times \mathbb{P}^1$ in Example 3.1, where the divisors corresponding to elements of $\Sigma(1)$ were denoted D_1, D_2, D_3, D_4. If the corresponding variables are x_1, x_2, x_3, x_4, then we get the ring $S = \mathbb{C}[x_1,x_2,x_3,x_4]$. This is graded by the Picard group, which is $\mathbb{Z}^2$ in this case. Using (3.3), we see that

$$\deg(x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}) = (a_1+a_2, a_3+a_4),$$

which is precisely the usual bigrading on $\mathbb{C}[x_1,x_2;x_3,x_4]$, where each graded piece consists of bihomogeneous polynomials in x_1, x_2 and x_3, x_4.

More generally, for $\mathbb{P}^n \times \mathbb{P}^m$, this construction gives $S = \mathbb{C}[x_0,\ldots,x_n;y_0,\ldots,y_m]$ with the usual bigrading. □

Example 3.4. For $\mathbb{P}(q_0,\ldots,q_n)$, the exact sequence (3.2) is described in Example 3.2. Here, the primitive generators of the rays are $v_0,\ldots,v_n$, corresponding to the divisors $D_0,\ldots,D_n$ and variables $x_0,\ldots,x_n$. Since $x_i = x^{D_i}$, we see that $\deg x_i = q_i$ since the map β is dot product with $(q_0,\ldots,q_n)$. Hence we recover the weighted grading of $\mathbb{C}[x_0,\ldots,x_n]$. □

We can also use the variables x_ρ to give coordinates on X_Σ. To do this, we need an analog of the "irrelevant" ideal $\langle x_0,\ldots,x_n\rangle \subset \mathbb{C}[x_0,\ldots,x_n]$. For each cone $\sigma \in \Sigma$, let $x^{\hat\sigma}$ be the monomial

$$x^{\hat\sigma} = \textstyle\prod_{\rho\not\subset\sigma} x_\rho,$$

and then define the irrelevant ideal $B \subset S$ to be

$$B = \big\langle x^{\hat\sigma} \mid \sigma \in \Sigma\big\rangle.$$

When Σ is the fan giving $\mathbb{P}^n$, the reader should check that $B = \langle x_0,\ldots,x_n\rangle$.

3.3. **The Quotient Construction.** The idea is that X_Σ should be a quotient of $\mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B)$, where $\mathbf{V}(B) \subset \mathbb{C}^{\Sigma(1)}$ is the variety of the irrelevant ideal B. The quotient is by the group G, which is defined to be

$$G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X_\Sigma), \mathbb{C}^*).$$

Note that applying $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)$ to (3.2) gives the exact sequence

$$1 \longrightarrow G \longrightarrow (\mathbb{C}^*)^{\Sigma(1)} \longrightarrow T(N) \tag{3.4}$$

since $T(N) = N \otimes_{\mathbb{Z}} \mathbb{C}^* = \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{C}^*)$. This shows that G acts naturally on $\mathbb{C}^{\Sigma(1)}$ and leaves $\mathbf{V}(B)$ invariant since this subvariety consists of coordinate subspaces.

The following representation of X_Σ was discovered independently by a variety of people (see [5] for references and a proof).

Theorem 3.1. *Assume that X_Σ is a toric variety such that $\Sigma(1)$ spans $N_\mathbb{R}$. Then:*
(1) *X_Σ is the universal categorical quotient $\big(\mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B)\big)/G$.*
(2) *X_Σ is a geometric quotient $\big(\mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B)\big)/G$ if and only if Σ is simplicial.*

The "geometric quotient" mentioned in part (2) is the algebro-geometric analog of the usual idea of quotient under a group action, where elements of the quotient correspond to orbits. The "universal categorical quotient is more subtle. In the case of an affine variety Spec(R), this quotient is Spec(R^G), where R^G is the ring of invariants under the action of G on R. We will see an example of this below.

Under the hypotheses of the theorem, one can *define* X_Σ to be the quotient $\big(\mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B)\big)/G$. To see why, note that (3.4) is short exact in this case, so that the torus $T(N)$ is the quotient $(\mathbb{C}^*)^{\Sigma(1)}/G$. Thus

$$T(N) = (\mathbb{C}^*)^{\Sigma(1)}/G \subset \big(\mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B)\big)/G.$$

Furthermore, since the "big" torus $(\mathbb{C}^*)^{\Sigma(1)}$ acts naturally on $\mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B)$, it follows that $T(N)$ acts on X_Σ. Quotients preserve normality, so that all of the requirements of being a normal toric variety are satisfied by the quotient in Theorem 3.1.

Before giving examples of Theorem 3.1, we explain how to compute $\mathbf{V}(B)$ and G explicitly:

- Ray generators $v_{\rho_1}, \ldots, v_{\rho_s}$ form a *primitive collection* (this terminology is due to Batyrev) if they don't lie in any cone of Σ but every proper subset does. Then one can show that
$$\mathbf{V}(B) = \bigcup_{v_{\rho_1},\ldots,v_{\rho_s} \text{ primitive}} \mathbf{V}(x_{\rho_1}, \ldots, x_{\rho_s}).$$
- If we write the ray generators as $v_1, \ldots, v_r$, $r = |\Sigma(1)|$, then
$$G = \{(\mu_1, \ldots, \mu_r) \mid \textstyle\prod_{i=1}^r \mu_i^{\langle m, v_i\rangle} = 1 \text{ for all } m \in M\}.$$
Hence if $m_1, \ldots, m_n$ is a $\mathbb{Z}$-basis of M, then $(\mu_1, \ldots, \mu_r) \in G$ if and only if
$$\textstyle\prod_{i=1}^r \mu_i^{\langle m_1, v_i\rangle} = \cdots = \prod_{i=1}^r \mu_i^{\langle m_n, v_i\rangle} = 1. \tag{3.5}$$

Here are the promised examples.

Example 3.5. Continuing our example of $\mathbb{P}^1 \times \mathbb{P}^1$, the reader should check that the irrelevant ideal is $B = \langle x_1x_3, x_1x_4, x_2x_3, x_2x_4\rangle$ and that $v_1 = e_1, v_2 = -e_1$ and $v_3 = e_2, v_4 = -e_2$ are the only primitive collections. Then, thinking of $\mathbb{C}^{\Sigma(1)}$ as $\mathbb{C}^2 \times \mathbb{C}^2$, one has

$$\mathbf{V}(B) = \mathbf{V}(x_1, x_2) \cup \mathbf{V}(x_3, x_4) = \{0\} \times \mathbb{C}^2 \cup \mathbb{C}^2 \times \{0\}.$$

To compute $(\mathbb{C}^*)^2 \simeq G \subset (\mathbb{C}^*)^4$, we note that by (3.5), $(\mu_1, \mu_2, \mu_3, \mu_4) \in G$ if and only if

$$\mu_1\mu_2^{-1} = \mu_3\mu_4^{-1} = 1.$$

Hence $G = \{(\mu, \mu, \lambda, \lambda) \mid \mu, \lambda \in \mathbb{C}^*\}$. Hence the quotient of Theorem 3.1 becomes

$$\big(\mathbb{C}^2 \times \mathbb{C}^2 \setminus (\{0\} \times \mathbb{C}^2 \cup \mathbb{C}^2 \times \{0\})\big)/(\mathbb{C}^*)^2,$$

which is exactly the way one usually represents $\mathbb{P}^1 \times \mathbb{P}^1$ as a quotient. □

Example 3.6. For $\mathbb{P}(q_0, \ldots, q_n)$, the irrelevant ideal is $\langle x_0, \ldots, x_n\rangle$ ($v_0, \ldots, v_n$ is the unique primitive collection!), so that $\mathbb{P}(q_0, \ldots, q_n)$ is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by $G \simeq \mathbb{C}^*$. However, the embedding $\mathbb{C}^* \to (\mathbb{C}^*)^{n+1}$ is determined by β, which in this case is the map $\mathbb{Z}^{n+1} \to \mathbb{Z}$ given by dot product with $(q_0, \ldots, q_n)$. It follows that

$$G = \{(\lambda^{q_0}, \ldots, \lambda^{q_n}) \mid \lambda \in \mathbb{C}^*\}.$$

Hence we recover the definition of $\mathbb{P}(q_0, \ldots, q_n)$ given in Section 0.11 of the BACKGROUND. □

Example 3.7. Let $e_1, \dots, e_n$ be a basis of $N = \mathbb{Z}^n$, and let σ be the cone they generate. The resulting affine toric variety is $\mathbb{C}^n$. The goal of this example is to construct global coordinates for the blow-up of $0 \in \mathbb{C}^n$. A first observation is that if $\partial\sigma$ is the fan consisting of all proper faces of σ, then $X_{\partial\sigma} = \mathbb{C}^n \setminus \{0\}$ since the n-dimensional cone σ corresponds to the fixed point 0.

Now let $e_0 = e_1 + \cdots + e_n$ and consider the fan Σ whose cones are generated by all proper subsets of $\{e_0, \dots, e_n\}$, excluding $\{e_1, \dots, e_n\}$. Let's first argue that X_Σ is the blow-up of $0 \in \mathbb{C}^n$. In Σ, consider the ray ρ_0 generated by e_0. This corresponds to a divisor $D_0 \subset X_\Sigma$. We can describe D_0 using the methods of Section 2.3 in LECTURE 2. The star of ρ_0 consists of all cones of Σ containing e_0. If we collapse ρ_0 to a point, we get a fan in an $(n-1)$-dimensional quotient of $\mathbb{R}^n$, which is easily seen to be the fan of $\mathbb{P}^{n-1}$. Thus $D_0 \simeq \mathbb{P}^{n-1}$. Furthermore, if we remove the star of ρ_0 from Σ, we are left with the fan $\partial\sigma$ from the previous paragraph. It follows that $X_\Sigma \setminus D_0 = \mathbb{C}^n \setminus \{0\}$. This makes it clear that we have the desired blow-up.

Let x_i correspond to the ray generated by e_i. Then the homogeneous coordinate ring of X_Σ is $\mathbb{C}[x_0, \dots, x_n]$ where $\deg(x_0) = -1$ and $\deg(x_i) = +1$ for $1 \le i \le n$. Furthermore, the only primitive collection is $e_1, \dots, e_n$, so that $\mathbf{V}(B) = \mathbb{C} \times \{0, \dots, 0\}$, and using (3.5), one sees that $(\mu_0, \dots, \mu_n) \in G$ if and only if

$$\mu_0\mu_1 = \mu_0\mu_2 = \cdots = \mu_0\mu_n = 1.$$

Hence $G = \{(\mu^{-1}, \mu, \dots, \mu) \mid \mu \in \mathbb{C}^*\} \simeq \mathbb{C}^*$, which acts on $\mathbb{C}^{\Sigma(1)} = \mathbb{C} \times \mathbb{C}^n$ by $\mu \cdot (x_0, \mathbf{x}) = (\mu^{-1}x_0, \mu\mathbf{x})$. Then, given $(x_0, \mathbf{x}) \in \mathbb{C} \times \mathbb{C}^n \setminus \mathbf{V}(B)$, we can act on this point using G to obtain

$$\begin{array}{lll} (x_0, \mathbf{x}) \sim_G (1, x_0\mathbf{x}) & \text{if } x_0 \neq 0 \\ (0, \mathbf{x}) \ \sim_G (0, \mu\mathbf{x}) & \text{if } \mu \neq 0. \end{array}$$

It is now easy to see that $X_\Sigma = \big(\mathbb{C} \times \mathbb{C}^n \setminus \mathbf{V}(B)\big)/G$ is the blow-up of $0 \in \mathbb{C}^n$. This approach gives global coordinates $x_0, x_1, \dots, x_n$ for the blow-up (subject to the action of $G \simeq \mathbb{C}^*$). In terms of these coordinates, the blow-up map $X_\Sigma \to \mathbb{C}^n$ is given by $(x_0, x_1, \dots, x_n) \mapsto (x_0x_1, \dots, x_0x_n)$ (note that x_0x_i has degree 0 and hence is invariant under the group action). □

Example 3.8. Let $\sigma \subset N_\mathbb{R} \simeq \mathbb{R}^n$ be an n-dimensional cone. Then the representation of V_σ given by Theorem 3.1 is of the form $\mathbb{C}^r/G$, where r is the number of rays of σ. This follows because $\mathbf{V}(B) = \emptyset$—a single cone has no primitive collections. Furthermore, there are two cases where G can be determined explicitly:

- For σ smooth, $G = \{1\}$, so that Theorem 3.1 gives $V_\sigma = \mathbb{C}^n$.
- For σ simplicial, $G \simeq N/N'$, $N' = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_{n+1}$ (you should check this carefully), so that by Theorem 3.1, V_σ is the quotient of $\mathbb{C}^n$ by the finite group G.

Note that the second bullet is precisely the representation given in (2.1) in LECTURE 2. □

Example 3.9. Let σ be the 3-dimensional cone of Example 1.8 from LECTURE 1. By Example 1.15 of LECTURE 1, we know that $V_\sigma = \mathbf{V}(xy - zw) \subset \mathbb{C}^4$. The ray generators $v_1 = e_1, v_2 = e_2, v_3 = e_1 + e_3, v_4 = e_2 + e_3$ of σ give variables x_1, x_2, x_3, x_4. It is straightforward to check that the group $G \subset (\mathbb{C}^*)^4$ consists of $(\lambda, \lambda^{-1}, \lambda^{-1}, \lambda)$ for $\lambda \in \mathbb{C}^*$ and that in the homogeneous coordinate ring $\mathbb{C}[x_1, x_2, x_3, x_4]$, the variables have degrees

$$\deg(x_1) = \deg(x_4) = 1,\ \deg(x_2) = \deg(x_3) = -1.$$

As in Example 3.8, V_σ is of the form $\mathbb{C}^4/G$, but this is not a geometric quotient since σ is not simplical. Rather, it is a "universal categorical quotient", which in this case means that

$$V_\sigma = \operatorname{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]^G).$$

One computes without difficulty that the ring of invariants is

$$\mathbb{C}[x_1, x_2, x_3, x_4]^G = \mathbb{C}[x_1x_2, x_3x_4, x_1x_3, x_2x_4].$$

This gives us two ways to think about V_σ:

- If m_1, m_2, m_3, m_4 are the generators of $\sigma^\vee$ from Example 1.15 of LECTURE 1, then we have an isomorphism
$$\mathbb{C}[x_1x_2, x_3x_4, x_1x_3, x_2x_4] \simeq \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^4]$$
defined by $x_1x_2 \mapsto t^{m_1}, x_3x_4 \mapsto t^{m_2}, x_1x_3 \mapsto t^{m_3}, x_2x_4 \mapsto t^{m_4}$. This shows that the ring of invariants really does give V_σ.
- The inclusion $\mathbb{C}[x_1, x_2, x_3, x_4]^G \subset \mathbb{C}[x_1, x_2, x_3, x_4]$ induces the quotient map $\pi : \mathbb{C}^4 \to \mathbb{C}^4/G = V_\sigma$ given by
$$\pi(x_1, x_2, x_3, x_4) = (x_1x_2, x_3x_4, x_1x_3, x_2x_4).$$
To see how this map differs from an ordinary quotient, let $p \in V_\sigma$. Then one can show that $\pi^{-1}(p)$ is a G-orbit when $p \neq (0,0,0,0)$, so that most of the time, the categorical quotient $\mathbb{C}^4/G$ behaves like an ordinary quotient. However, when $p = (0,0,0,0)$, one sees that
$$\pi^{-1}(p) = (\mathbb{C} \times \{0\} \times \{0\} \times \mathbb{C}) \cup (\{0\} \times \mathbb{C} \times \mathbb{C} \times \{0\}).$$
This shows that the stuff mapping to $(0,0,0,0)$ has dimension 2 and hence consists of infinitely many G-orbits. □

3.4. **Affine Toric Ideals and Monomial Maps.** Our second construction of toric varieties uses toric ideals. Let's begin with the affine case. Suppose that we have
$$\mathcal{A} = \{m_1, \dots, m_s\} \subset \mathbb{Z}^n.$$
This gives Laurent monomials $t^{m_1}, \dots, t^{m_s}$, and the map sending $y_i \mapsto t^{m_i}$ gives a homomorphism $\mathbb{C}[y_1, \dots, y_s] \to \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. The kernel of this map is the *toric ideal* $I_\mathcal{A}$.

Example 3.10. Let $\mathcal{A} = \{m_1 = e_1, m_2 = e_3, m_3 = e_3, m_4 = e_1 + e_2 - e_3\} \subset \mathbb{Z}^3$. In Example 1.15 of LECTURE 1, we saw that these are the minimal generators of $\sigma^\vee \cap \mathbb{Z}^3$, where σ is the 3-dimensional cone generated by $e_1, e_2, e_1 + e_3, e_2 + e_3$. We also observed that the map $x \mapsto t^{m_1}, y \mapsto t^{m_2}, z \mapsto t^{m_3}, w \mapsto t^{m_4}$ has kernel
$$\langle xy - zw \rangle \subset \mathbb{C}[x, y, z, w].$$
This is the toric ideal $I_\mathcal{A}$. □

In Example 3.10, the toric ideal is generated by *binomials* (a binomial is a difference to two monomials). This is true for all toric ideals. To describe this precisely, note that each $\alpha = (a_1, \dots, a_s) \in \mathbb{Z}^s$ can be uniquely written $\alpha = \alpha^+ - \alpha^-$, where α^+ and α^- have nonnegative entries and disjoint support. Then one can prove that the toric ideal $I_\mathcal{A} \subset \mathbb{C}[y_1, \dots, y_s]$ is given by
$$I_\mathcal{A} = \big\langle y^{\alpha^+} - y^{\alpha^-} \mid \alpha = (a_1, \dots, a_s) \in \mathbb{Z}^s,\ \textstyle\sum_{i=1}^s a_i m_i = 0 \big\rangle. \tag{3.6}$$
A proof of this assertion can be found in [34, Cor. 4.3]. Notice also that the definition of $I_\mathcal{A}$ gives an injection
$$\mathbb{C}[y_1, \dots, y_s]/I_\mathcal{A} \longrightarrow \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}].$$
Since the ring on the right is an integral domain, it follows that toric ideals are always prime.

Thinking geometrically, the ideal $I_\mathcal{A} \subset \mathbb{C}[y_1, \dots, y_s]$ defines an irreducible affine variety $V_\mathcal{A} \subset \mathbb{C}^s$. One can show that $V_\mathcal{A}$ is the Zariski closure of the image of the map $(\mathbb{C}^*)^n \to \mathbb{C}^s$ defined by
$$t \mapsto (t^{m_1}, \dots, t^{m_s}). \tag{3.7}$$
Note also that $V_\mathcal{A}$ contains a torus (the image of $(\mathbb{C}^*)^n$ under the map (3.7)), which is Zariski dense by the definition of $V_\mathcal{A}$. Furthermore, the proof of Theorem 1.13 from LECTURE 1 shows that the action of this torus on itself extends to an action on $V_\mathcal{A}$. It follows that $V_\mathcal{A}$ is an affine toric variety, though it need not be normal.

Example 3.11. For the subset $\mathcal{A} \subset \mathbb{Z}^3$ from Example 3.10, $V_\mathcal{A}$ is the normal affine toric variety V_σ, where $\sigma = \mathrm{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3)$ is the cone featured in many of our examples. □

Example 3.12. Given $\mathcal{A} = \{\beta_1, \dots, \beta_s\} \subset \mathbb{Z}$, we get a *monomial curve* in $\mathbb{C}^s$ parametrized by

$$t \mapsto (t^{\beta_1}, \dots, t^{\beta_s}).$$

Since t^{β_i} is a character on $\mathbb{C}^*$, this gives the affine toric variety $V_\mathcal{A}$. Since $V_\mathcal{A}$ has dimension 1, it is non-normal precisely when it fails to be smooth. The simplest example is the cusp parametrized by $t \mapsto (t^2, t^3)$. Here, the corresponding toric ideal is generated by the binomial $y^2 - x^3$. □

Example 3.13. Consider the surface in $\mathbb{C}^4$ parametrized by

$$(t, u) \mapsto (t^4, t^3u, tu^3, u^4).$$

The Zariski closure of the image of this map is $V_\mathcal{A}$ for

$$\mathcal{A} = \{(4,0), (3,1), (1,3), (0,4)\} \subset \mathbb{Z}^2.$$

By Exercise 3.18 of [21, Chapter I], $V_\mathcal{A}$ is not normal. One can also show that $V_\mathcal{A}$ is not Cohen-Macaulay (see [33]). □

Projectively, the parametrization given in Example 3.13 defines a twisted quartic curve $C \subset \mathbb{P}^3$, which is known to be normal. Hence C is normal but not projectively normal (since projective normality is equivalent to normality of the affine cone). A surprising number of basic examples in algebraic geometry are toric varieties in disguise.

According to [34, Prop. 13.5], $V_\mathcal{A}$ is a normal toric variety if and only if

$$\mathbb{N}\mathcal{A} = \mathrm{Cone}(\mathcal{A}) \cap \mathbb{Z}\mathcal{A},$$

where $\mathbb{Z}\mathcal{A}$ (resp. $\mathbb{N}\mathcal{A}$) is the set of all integer (resp. nonnegative integer) combinations of elements of $\mathcal{A}$. More generally, the normalization of $V_\mathcal{A}$ is the affine toric variety V_σ, where $\sigma \subset N_\mathbb{R}$ is the cone dual to $\mathrm{Cone}(\mathcal{A})$ and N is the dual of $\mathbb{Z}\mathcal{A}$.

We also note that in the literature (see [32], for example), affine toric ideals are often described in terms of an $n \times s$ matrix $\mathbf{A}$ with integer entries. The s columns of such a matrix give the subset $\mathcal{A} \subset \mathbb{Z}^n$ used above. There are two advantages to this approach:

- The description of the toric ideal $I_\mathcal{A} \subset \mathbb{C}[y_1, \dots, y_s]$ given in (3.6) can be rewritten as

$$I_\mathcal{A} = \langle y^a - y^b \mid a, b \in \mathbb{N}^s, \mathbf{A}a = \mathbf{A}b \rangle. \tag{3.8}$$

- The map $(\mathbb{C}^*)^n \to (\mathbb{C}^*)^s \subset \mathbb{C}^s$ defined in (3.7) is obtained by applying $\mathrm{Hom}_\mathbb{Z}(-, \mathbb{C}^*)$ to the map $\mathbb{Z}^s \to \mathbb{Z}^n$ given by matrix multiplication by $\mathbf{A}$.

3.5. **Projective Toric Ideals.** Let $\mathcal{A} = \{m_1, \dots, m_s\} \subset \mathbb{Z}^n$ be as above. Besides the affine toric variety $V_\mathcal{A} \subset \mathbb{C}^s$, we also get a projective toric variety $Y_\mathcal{A} \subset \mathbb{P}^{s-1}$ by regarding (3.7) as a map $(\mathbb{C}^*)^n \to \mathbb{P}^{s-1}$. More precisely, $Y_\mathcal{A}$ is defined to be the Zariski closure of the image of this map.

The ideal of $Y_\mathcal{A}$ consists of all *homogeneous* binomials $y^{\alpha^+} - y^{\alpha^-}$ as in (3.6). One nice case is when the $n \times s$ matrix $\mathbf{A}$ built from $\mathcal{A}$ contains the vector $(1, \dots, 1)$ in its row space. When this happens, the binomials $y^{\alpha^+} - y^{\alpha^-}$ of (3.6) are automatically homogeneous, so that the toric ideal $I_\mathcal{A}$ is the homogeneous ideal of $Y_\mathcal{A}$.

Example 3.14. As in Example 3.10, let $\mathcal{A} = \{m_1 = e_1, m_2 = e_3, m_3 = e_3, m_4 = e_1 + e_2 - e_3\} \subset \mathbb{Z}^3$. This gives the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

The sum of the rows is $(1, 1, 1, 1)$, so that the toric ideal is homogeneous. This is consistent with $I_{\mathcal{A}} = \langle xy - zw \rangle \subset \mathbb{C}[x, y, z, w]$, which is the homogeneous ideal of the surface $xy = zw$ in $\mathbb{P}^3$. □

One way to arrange for $\mathbf{A}$ to contain $(1, \ldots, 1)$ in its row space is to replace $\mathcal{A}$ with

$$\mathcal{A}^+ = \{(1, m_1), \ldots, (1, m_s)\} \subset \mathbb{Z}^{n+1}.$$

This gives the matrix $\mathbf{A}^+$ obtained from $\mathbf{A}$ by adding a row of 1's at the top. Then one can easily show the affine cone of $Y_{\mathcal{A}} \subset \mathbb{P}^{s-1}$ is the variety $V_{\mathcal{A}^+} \subset \mathbb{C}^s$. Hence the toric ideal $I_{\mathcal{A}^+}$ is homogeneous and defines the projective variety $Y_{\mathcal{A}}$.

Here is one interesting situation in which the projective toric variety $Y_{\mathcal{A}}$ arises naturally.

Example 3.15. Suppose that $\mathcal{A} = \{m_1, \ldots, m_s\} \subset \mathbb{Z}^n$ and that $\mathcal{A}$ generates $\mathbb{Z}^n$. Then let $L(\mathcal{A})$ be the set of Laurent polynomials with exponent vectors in $\mathcal{A}$, i.e.,

$$L(\mathcal{A}) = \big\{a_1 t^{m_1} + \cdots + a_s t^{m_s} \mid a_i \in \mathbb{C}\big\}.$$

Given $n + 1$ Laurent polynomials $f_0, \ldots, f_n \in L(\mathcal{A})$, their $\mathcal{A}$-*resultant*

$$\mathrm{Res}_{\mathcal{A}}(f_0, \ldots, f_n)$$

is a polynomial in the coefficients of the f_i whose vanishing is necessary and sufficient for the equations $f_0 = \cdots = f_n = 0$ to "have a solution" (see [18, Prop. 2.1]). However, one must be careful where the solution lies. The f_i are defined initially on the torus $(\mathbb{C}^*)^n$, but the definition of $Y_{\mathcal{A}}$ shows that the equation $f_i = 0$ makes sense on $Y_{\mathcal{A}}$. Then one can prove that

$$\mathrm{Res}_{\mathcal{A}}(f_0, \ldots, f_n) = 0 \iff f_1 = \cdots = f_n = 0 \text{ have a solution in } Y_{\mathcal{A}}.$$

The relation between toric varieties and resultants is described in [10, 18].

Basic references for toric ideals and non-normal toric varieties are [18, 34]. Also, some applications to combinatorics can be found in [34, Ch. 14].

3.6. **Lattice Ideals.** We conclude this lecture by pointing out that toric ideals can be generalized as follows. Let $L \subset \mathbb{Z}^s$ be a subgroup, which we call a lattice since it is free Abelian. Then the *lattice ideal* of L is defined by

$$I_L = \big\langle y^a - y^b \mid a, b \in \mathbb{N}^s, a - b \in L \big\rangle \subset \mathbb{C}[y_1, \ldots, y_s].$$

Example 3.16. Let $L = \ker \mathbf{A}$, where the $n \times s$ matrix $\mathbf{A}$ comes from $\mathcal{A} \subset \mathbb{Z}^n$. Then $a - b \in L$ if and only if $\mathbf{A}a = \mathbf{A}b$. By (3.8), it follows that I_L is the toric ideal $I_{\mathcal{A}}$. □

We showed in Section 3.4 that toric ideals are always prime. By Theorem 7.4 of [25], a lattice ideal is toric if and only if it is prime.

Here is an example (taken from [25]) of a lattice ideal that is not prime and hence not toric.

Example 3.17. Let $L = \{(a, b, c) \in \mathbb{Z}^3 \mid a + b + c \equiv 0 \bmod 2\}$. Then one can compute that

$$I_L = \langle x^2 - 1, xy - 1, yz - 1 \rangle.$$

Since $\mathbf{V}(I_L) = \{\pm(1, 1, 1)\}$, it is clear that I_L is not prime. In fact,

$$I_L = \langle x - 1, y - 1, z - 1 \rangle \cap \langle x + 1, y + 1, z + 1 \rangle.$$

This intersection of maximal ideals is the primary decomposition of I_L. □

Much more material on lattice ideals can be found in [25].

LECTURE 4. POLYTOPES AND TORIC VARIETIES

This lecture will explore the deep relation between toric varieties and poltytopes.

4.1. **Lattice Polytopes.** Let $N \simeq \mathbb{Z}^n$ be a lattice with dual M. A *lattice polytope* $P \subset M_{\mathbb{R}} \simeq \mathbb{R}^n$ is the convex hull of a finite subset of M. In this lecture, we will always assume that P has dimension n.

The definition of a *face* of a polytope P is similar to the definition of face of a cone—we leave the definition to the reader. A *facet* is a face of codimension 1 and a *vertex* is a face of dimension 0. Note that the vertices of a lattice polytope lie in the lattice M. Our assumption that $\dim P = n$ implies that the normal vector to a facet F of P is unique up to multiplication by a nonzero real number. Since the facet is defined over M, we can pick a unique *facet normal* $n_F \in N$ by requiring that n_F be primitive and point toward the interior of P .

Every lattice polytope has two representations, one as a convex hull as above, and the other as an intersection of closed halfspaces. Each facet F of P has a supporting hyperplane defined by

$$\langle m, n_F \rangle = -a_F$$

for some $a_F \in \mathbb{Z}$. Then the polytope is given by

$$P = \bigcap_{F \text{ is a facet}} \{m \in M_{\mathbb{R}} \mid \langle m, n_F \rangle \geq -a_F\}. \tag{4.1}$$

4.2. **Normal Fans.** Given P as above and a face $\mathcal{F}$ of P (not necessarily a facet), let $\sigma_{\mathcal{F}}$ be the cone in $N_{\mathbb{R}}$ generated by the facet normals n_F for all facets F containing $\mathcal{F}$. It is easy to see that $\sigma_{\mathcal{F}}$ is a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ and that the set of cones

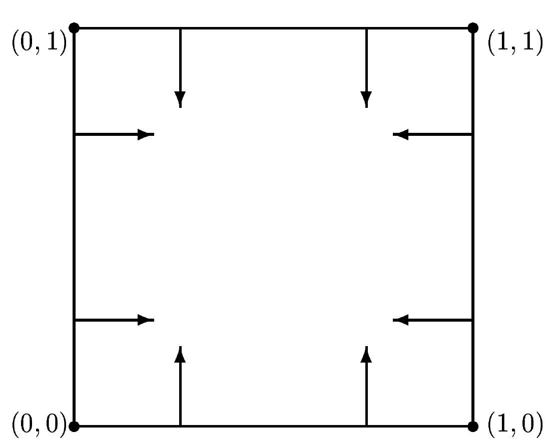
$$\Sigma_P = \{\sigma_{\mathcal{F}} \mid \mathcal{F} \text{ is a face of } P\}$$

is a fan, called the *normal fan* of P. This gives a toric variety denoted X_P.

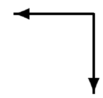
Example 4.1. The unit square $\square$ with vertices $(0,0), (1,0), (1,1), (0,1)$ can be represented

$$\square = \{a \geq 0\} \cap \{b \geq 0\} \cap \{-a \geq -1\} \cap \{-b \geq -1\}.$$

It follows that the inward normals are $\pm e_1$ and $\pm e_2$ in $\mathbb{Z}^2$. These can be pictured as follows (not drawn to scale):



Each inward normal appears twice to show that each vertex gives a 2-dimensional cone in the normal fan. For example, the vertex $(1, 1)$ gives the 2-dimensional cone



The other vertices are handled similarly, and the resulting normal fan is the one appearing in on the right in Example 2.2 of LECTURE 2. Hence $X_\square = \mathbb{P}^1 \times \mathbb{P}^1$. □

Exercise 4.1. Show that the normal fan of the lattice polygon $P = \mathrm{Conv}(0, 2e_1, e_2) \subset \mathbb{R}^2$ is the fan pictured in Example 2.4 of LECTURE 2. Hence $X_P = \mathbb{P}(1, 1, 2)$. □

In general, we can characterize normal fans as follows.

Theorem 4.1. *The normal toric variety of a fan Σ in $N_\mathbb{R} \simeq \mathbb{R}^n$ is projective if and only if Σ is the normal fan of an n-dimensional lattice polytope in $M_\mathbb{R}$.*

A useful observation is that the polytope P is combinatorially dual to its normal fan Σ_P. This means that there is a one-to-one inclusion reversing correspondence

$$\sigma_\mathcal{F} \in \Sigma_P \longleftrightarrow \mathcal{F} \subset P$$

between cones of Σ_P and faces of P (provided we count P as a face of itself) such that

$$\dim \sigma_\mathcal{F} + \dim \mathcal{F} = n \tag{4.2}$$

for all faces $\mathcal{F}$ of P. Combining this with the correspondence between cones in Σ_P and torus orbits in X_P from Section 2.3 from LECTURE 2, we get a one-to-one dimension preserving correspondence between faces of P and torus orbits of X_P. Thus:

- Vertices of $P \longleftrightarrow n$-dimensional cones of $\Sigma_P \longleftrightarrow$ fixed points of the torus action in X_P.
- Facets of $P \longleftrightarrow$ rays of $\Sigma_P \longleftrightarrow$ torus-invariant irreducible divisors in X_P.

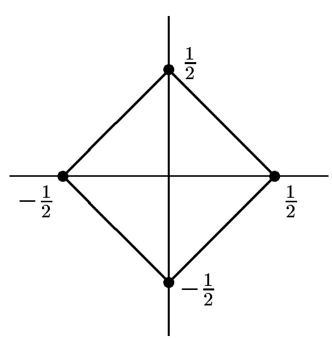
In general, P determines the combinatorics of the toric variety X_P.

There is also a dual construction of X_P when P contains the origin as an interior point. In this situation, the *polar* or *dual* of $P \subset M_\mathbb{R}$ is defined by

$$P^\circ = \{u \in N_\mathbb{R} \mid \langle m, u\rangle \geq -1 \text{ for all } m \in P\}.$$

One can show that P° is an n-dimensional polytope containing the origin in its interior. The vertices of P° lie in $M_\mathbb{Q} = M \otimes_\mathbb{Z} \mathbb{Q}$, though P° need not be a lattice polytope.

Example 4.2. The lattice polytope $P = \mathrm{Conv}(\pm 2e_1 \pm 2e_2) \subset \mathbb{R}^2$ contains the origin. Its polar P° is a tilted square in the plane:



Note that P° is not a lattice polytope. □

Given $P \subset M_{\mathbb{R}}$ and $P^\circ \subset N_{\mathbb{R}}$ as above, one can show that the normal fan Σ_P in $N_{\mathbb{R}}$ by taking cones (relative to the origin) over the faces of P.

Example 4.3. The fan obtained by taking cones over faces of the polytope $P^\circ \subset \mathbb{R}^2$ of Example 4.2 is clearly the fan on the right in Example 2.2 from LECTURE 2 that gives $\mathbb{P}^1 \times \mathbb{P}^1$. □

4.3. **Ample Divisors and Monomial Maps.** Let $P \subset M_{\mathbb{R}} \simeq \mathbb{R}^n$ be an n-dimensional lattice polytope. The facet normals of P determine the normal fan Σ_P that gives the toric variety X_P. But P contains more data than just its facet normals n_F, for the representation (4.1) also includes the integers a_F appearing in the defining equations

$$\langle m, n_F\rangle = -a_F$$

of the supporting hyperplanes of the facets. Each facet F corresponds to a torus-invariant divior D_F on X_P, so that P determines the divisor

$$D_P = \sum_F a_F D_F,$$

where the sum is over all facets of P.

Example 4.4. Given positive integers r and s, let $P_{r,s} \subset \mathbb{R}^2$ be the $r \times s$ rectangle with vertices $(0,0), (r,0), (0,s), (r,s)$. In terms of (4.1),

$$P_{r,s} = \{a \geq 0\} \cap \{b \geq 0\} \cap \{-a \geq -r\} \cap \{-b \geq -s\}.$$

This has facet normals $n_1 = e_1, n_2 = e_2, n_3 = -e_1, n_4 = -e_2$. The normal fan is the same as for Example 4.1, so that $X_{P_{r,s}} = \mathbb{P}^1 \times \mathbb{P}^1$. If D_1, D_2, D_3, D_4 are the divisors corresponding to n_1, n_2, n_3, n_4, then

$$D_{P_{r,s}} = rD_3 + sD_4$$

since $a_1 = a_2 = 0, a_3 = r, a_4 = s$. □

In general, the divisor D_P has some important properties. We begin with an easy one.

Proposition 4.2. *D_P is a Cartier divisor on X_P.*

Proof. Let m be a vertex of P and σ be the corresponding n-dimensional cone of Σ_P. This gives the affine open subset $V_\sigma \subset X_\Sigma$. We leave it as an exercise to check that for a facet F of P,

$$D_F \cap V_\sigma \neq 0 \iff F \text{ contains } m.$$

Using the character χ^m is a rational function on X_P, we obtain

$$\operatorname{div}(\chi^m)\big|_{V_\sigma} = \sum_{m \in F} \langle m, n_F\rangle D_F = -\sum_{m\in F} a_F D_F = -D_P|_{V_\sigma},$$

where the first equality is by (3.1) from LECTURE 3, the second follows since $\langle m, n_F\rangle = -a_F$ for all facets containing m, and the third follows from the definition of D. This shows that D is locally principal and hence Cartier (see Section 0.16 from the BACKGROUND). □

A deeper property of D_P is that it is *ample*. We can explain this as follows. Define

$$H^0(X_P, \mathcal{O}_{X_P}(D_P)) = \{f \in \mathbb{C}(X_P)^* \mid \operatorname{div}(f) + D_P \geq 0\}. \tag{4.3}$$

Here, $\mathcal{O}_{X_P}(D_P)$ is the invertible sheaf associated to the Cartier divisor D_P. We won't discuss sheaves in these lectures—see [15, 21] for an introduction to sheaf theory. Recall from Section 0.15 in the BACKGROUND that $\operatorname{div}(f) + D_P \geq 0$ means that the divisor is *effective*, i.e., is a nonnegative linear combination of irreducible divisors.

The remarkable fact is that the rational functions f appearing in (4.3) are determined by the lattice points of P.

Proposition 4.3. $H^0(X_P, \mathcal{O}_{X_P}(D_P)) = \bigoplus_{m \in P \cap M} \mathbb{C} \cdot \chi^m$.

Proof. Since D_P is torus-invariant, the torus of X_P acts on $H^0(X_P, \mathcal{O}_{X_P}(D_P))$. It follows (see [17, Sec. 3.3] for details) that this space spanned by characters. Using (3.1) from LECTURE 3, we see that $\operatorname{div}(\chi^m) + D_P \geq 0$ if and only if

$$\sum_F \langle m, n_F \rangle D_F + \sum_F a_F D_F \geq 0.$$

This is equivalent to $\langle m, n_F \rangle \geq -a_F$ for all F, which by (4.1) means that $m \in P \cap M$. □

Divisors are used in algebraic geometry to give maps to projective space. For D_P, one uses the rational functions in $H^0(X_P, \mathcal{O}_{X_P}(D_P))$, which by Proposition 4.3 means the characters coming from lattice points of P. To write this concretely, assume that $N = M = \mathbb{Z}^n$, so that a lattice point $m \in P \cap \mathbb{Z}^n$ gives the character $\chi^m = t^m$ on the torus $(\mathbb{C}^*)^n$ of X_P. Let $m_1, \ldots, m_\ell$ be the lattice points of P. Then we get the map

$$\varphi(t_1, \ldots, t_n) = \big(t^{m_1}, \ldots, t^{m_\ell}\big) \in \mathbb{P}^{\ell-1}, \quad \ell = |P \cap \mathbb{Z}^n|, \tag{4.4}$$

from $(\mathbb{C}^*)^n$ to $\mathbb{P}^{\ell-1}$. This is a special case of (3.7) from LECTURE 3 and gives the projective toric variety $Y_{P\cap\mathbb{Z}^n}$ defined in Section 3.5 of LECTURE 3.

The toric varieties X_P and $Y_{P\cap\mathbb{Z}^n}$ are closely related. To understand this, we first note that if ν is a positive integer, then P and νP have the same normal fan and toric variety. Thus $X_P = X_{\nu P}$. Furthermore, it is easy to see that the divisor associated to νP is $D_{\nu P} = \nu D_P$.

Then $H^0(X_P, \mathcal{O}_{X_P}(\nu D_P))$ gives the map

$$\varphi_\nu : (\mathbb{C}^*)^n \longrightarrow \mathbb{P}^{\ell_\nu - 1}, \quad \ell_\nu = |(\nu P) \cap \mathbb{Z}^n|,$$

defined by the lattice points of νP. For $\nu \gg 0$, one can prove that φ_ν induces an isomorphism

$$X_P \simeq Y_{(\nu P)\cap\mathbb{Z}^n}. \tag{4.5}$$

In other words, $H^0(X_P, \mathcal{O}_{X_P}(\nu D_P))$ gives a projective embedding of X_P for $\nu \gg 0$. This is what it means to say that the divisor D_P is *ample*. One consequence of (4.5) is that X_P is a projective variety, as claimed in Theorem 4.1.

Finally, we note that while (4.4) does not give a projective embedding of X_P in general, it is known to give an embedding in the following two cases:

- X_P is smooth, or
- we replace P with $(n-1)P$, $n = \dim P$.

It follows that when P is a polygon, we have $(n-1)P = (2-1)P = P$. Thus (4.4) always gives an embedding when X_P is a toric surface. More generally, the second bullet shows that the above $\nu \gg 0$ becomes $\nu \geq n-1$, so that (4.5) is an isomorphism for $\nu \geq n-1$. This gives an elementary way to construct the toric variety X_P: given P, take the projective toric variety $Y_{(\nu P)\cap M}$ parametrized by the Laurent monomials coming from the lattice points of νP, where ν is any integer $\geq n-1$.

Even though (4.4) need not give an embedding of X_P, it always extends to an everywhere defined map $X_P \to Y_{P\cap M} \subset \mathbb{P}^{\ell-1}$ (we will prove this below). Furthermore, the map $X_P \to Y_{P\cap M}$ has the following explicit description. Let $\widetilde{M} \subset M$ be the sublattice generated by the differences $m - m'$ for all $m, m' \in P \cap \mathbb{Z}^n$. Dualizing, we get a lattice $\widetilde{N} \supset N$. Then $X_P \to Y_{P\cap M}$ factors

$$X_P = X_{\Sigma_P, N} \longrightarrow X_{\Sigma_P, \widetilde{N}} \longrightarrow Y_{P\cap M} \tag{4.6}$$

where the first map is the quotient of X_P by the finite group $\widetilde{N}/N$ (by Section 2.7 of LECTURE 2) and the second map is the normalization map (by the discussion following Theorem 13.12 of [34]).

Example 4.5. Let $P = \mathrm{Conv}(0, e_1, e_2, -e_1+e_2+2e_3) \subset M_{\mathbb{R}} = \mathbb{R}^3$, where $M = \mathbb{Z}^3$. Then $\widetilde{M}$ equals $\mathbb{Z}e_1 + \mathbb{Z}e_2 + 2\mathbb{Z}e_3$, so that $\widetilde{M}$ has index 2 in M. Hence $\widetilde{N} = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \frac{1}{2}\mathbb{Z}e_3$, so that $N = \mathbb{Z}^3$ is a sublattice of index 2 in $\widetilde{N}$. By (4.6), X_P is a 2-to-1 cover of the normalization of $Y_{P\cap M}$.

For completeness, let's determine the toric varieties X_P, $X_{\Sigma_P,\widetilde{N}}$, and $Y_{P\cap M}$. The latter is easy, since one computes without difficulty that $P \cap M$ consists of the vertices of P. It follows that $Y_{P\cap M} = \mathbb{P}^3$. As for the other two, the facet normals of P relative to N are

$$n_1 = e_3,\ n_2 = 2e_1 + e_3,\ n_3 = 2e_2 - e_3,\ n_4 = -2e_1 - 2e_2 - e_3.$$

These were computed using the program `polymake`, which can be downloaded from the internet (google "polymake"). Note that $n_1 + n_2 + n_3 + n_4 = 0$.

If we switch to $\widetilde{N}$, the normals are $\widetilde{n}_i = \frac{1}{2}n_i$, and it easy easy to see that they generate $\widetilde{N}$. Since they sum to 0, we see that $X_{\Sigma_P,\widetilde{N}} = \mathbb{P}^3$. This of course is consistent with $Y_{P\cap M} = \mathbb{P}^3$ since $X_{\Sigma_P,\widetilde{N}} \to Y_{P\cap M}$ is the normalization map.

Finally, the original facet normals n_i generate the sublattice $N' = 2\mathbb{Z}e_1 + 2\mathbb{Z}e_2 + \mathbb{Z}e_3 \subset N$. Relative to N', we clearly have $X_{\Sigma_P,N'} = \mathbb{P}^3$, so that by Section 2.7 of Lecture 2, X_P is the quotient of $\mathbb{P}^3$ by $N/N' \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. □

4.4. **Homogeneous Coordinates.** For the toric variety X_P, the homogeneous coordinate ring defined in Lecture 3 has a nice description as follows. By (4.2), 1-dimensional cones of the normal fan correspond to facets of P. It follows that variables correspond to facets. We label the facets as $F_1, \ldots, F_r$ and the facet normals as $n_1, \ldots, n_r$. Accordingly, the variables in the homogeneous coordinate ring will be labeled as $x_1, \ldots, x_r$. These are the *facet variables* of the polytope P.

The irrelevant ideal $B = \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma_P\rangle \subset \mathbb{C}[x_1, \ldots, x_r]$ defined in Section 3.2 from Lecture 3 also has a nice description. First observe that B is generated by the monomials $x^{\hat{\sigma}}$ coming from n-dimension cones in the normal fan, which correspond to vertices P. It follows easily that if σ corresponds to the vertex $m \in P$, then $x^{\hat{\sigma}}$ is the product of those variables whose facets miss the vertex v. So the generators of B can be determined directly from the polytope.

The lattice points of P give some interesting monomials in the homogeneous coordinate ring $\mathbb{C}[x_1, \ldots, x_r]$. Write (4.1) as

$$P = \bigcap_i \{m \in M_{\mathbb{R}} \mid \langle m, n_i\rangle \geq -a_i\}. \tag{4.7}$$

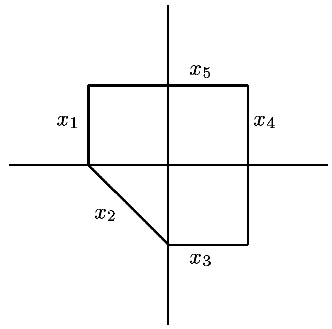
Then, given $m \in P \cap M$, set

$$x^{[m]} = \prod_{i=1}^{r} x_i^{\langle m,n_i\rangle + a_i}.$$

We call $x^{[m]}$ a *P-monomial.* The description (4.7) of P shows that the exponents of $x^{[m]}$ are all ≥ 0, so that x^m is in the homogeneous coordinate ring.

One nice observation is that the exponent of x_i in $x^{[m]}$ gives the *lattice distance* from m to the facet F_i. To see this, suppose that the exponent of x_i is $a > 0$. The facet F_i lies in the hyperplane $\{m \in \mathbb{R}^n \mid \langle m, n_i\rangle + a_i = 0\}$. To get from here to m, we must pass through the a parallel hyperplanes, namely $\{m \in \mathbb{R}^n \mid \langle m, n_i\rangle + a_i = j\}$ for $j = 1, \ldots, a$. Here is an example.

Example 4.6. Consider the toric variety X_P of the polytope $P \subset \mathbb{R}^2$, with facet variables as indicated:



Note that P has vertices $(1,1),(-1,1),(-1,0),(0,-1),(1,-1)$. In terms of (4.7), we have $a_1 = \cdots = a_5 = 1$, where the indices correspond to $x_1,\ldots,x_5$. The 8 points of $P\cap\mathbb{Z}^2$ give the following P-monomials:

$$\begin{array}{ccc} x_2x_3^2x_4^2, & x_1x_2^2x_3^2x_4, & x_1^2x_2^3x_3^2 \\ x_3x_4^2x_5, & x_1x_2x_3x_4x_5, & x_1^2x_2^2x_3x_5 \\ & x_1x_4x_5^2, & x_1^2x_2x_5^2. \end{array}$$

In this display, the position of each P-monomial $x^{[m]}$ corresponds to the position of the lattice point $m\in P\cap\mathbb{Z}^2$. □

Recall that the degree of a monomial $\prod_{i=1}^r x_i^{\alpha_i}$ is the class of the divisor $\sum_{i=1}^r \alpha_i D_i$ in the Chow group $A_{n-1}(X_P)$ (see (3.2) from LECTURE 3). A nice property of P-monomials is that they all have the same degree. To see this, let $G = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X_P),\mathbb{C}^*)$. For $\mu = (\mu_1,\ldots,\mu_r)\in G$, we set

$$\mu^P = \prod_i \mu_i^{a_i}.$$

Then, given $m\in P\cap\mathbb{Z}^n$, we have

$$\mu\cdot x^{[m]} = \prod_{i=1}^r (\mu_i x_i)^{\langle m,n_i\rangle + a_i} = \mu^P\, x^{[m]} \tag{4.8}$$

since $\prod_{i=1}^r \mu_i^{\langle m,n_i\rangle} = 1$ by the description of G given in Section 3.3 of LECTURE 3. It follows that all P-monomials transform the same way under G, which means that they have the same degree. Furthermore, one can show the P-monomials give *all* monomials of this degree. Comparing this to Proposition 4.3, we see that the graded piece of $\mathbb{C}[x_1,\ldots,x_r]$ in this degree is naturally isomorphic to $H^0(X_P,\mathcal{O}_{X_P}(D_P))$.

Here are two exercises that explore other aspects of P-monomials.

Exercise 4.2. Prove that the lattice points in the interior $\mathrm{int}(P)$ of P correspond precisely to those P-monomials which are divisible by $x_1\cdots x_r$.

Exercise 4.3. A P-monomial corresponding to a vertex of P is called a *vertex monomial* and let $B' = \langle x^{[m]} \mid m \text{ is a vertex of } P\rangle$ be the monomial ideal generated by the vertex monomials. Prove that $\sqrt{B'} = B$, where B is the irrelevant ideal defined above. This shows that $\mathbf{V}(B)\subset\mathbb{P}^r$ is the subvariety defined by the vanishing of the vertex monomials.

We can also use P-monomials to give a homogeneous version of the map (4.4) which uses the quotient representation $X_P = (\mathbb{C}^r\setminus\mathbf{V}(B))/G$. As in Section 4.3, let $m_i, i=1,\ldots,\ell$ be the lattice points of $P\cap\mathbb{Z}^n$. Then consider the map

$$(x_1,\ldots,x_r)\longrightarrow (x^{[m_1]},\ldots,x^{[m_\ell]}). \tag{4.9}$$

First observe that if m_j is a vertex of P, then $x^{[m_j]}$ is a vertex monomial. Since $\mathbf{V}(B)$ is defined by the vanishing of these monomials (Exercise 4.3), it follows that (4.9) gives a well-defined map

$$\phi : \mathbb{C}^r \setminus \mathbf{V}(B) \longrightarrow \mathbb{P}^{\ell-1}.$$

Furthermore, given $x \in \mathbb{C}^r \setminus \mathbf{V}(B)$ and $\mu \in G$, (4.8) implies that

$$\phi(\mu \cdot x) = \mu^P \phi(x).$$

Since we are mapping to projective space, ϕ induces a well-defined map

$$X_P = \big(\mathbb{C}^r \setminus \mathbf{V}(B)\big)/G \longrightarrow \mathbb{P}^{\ell-1}. \tag{4.10}$$

The surprise is that if one restricts this map to $(\mathbb{C}^*)^n \subset X_P$, then one gets *exactly* the map (4.4)

$$\varphi(t_1, \ldots, t_n) = (t^{m_1}, \ldots, t^{m_\ell}) \in \mathbb{P}^{\ell-1}$$

defined by the Laurent monomials of lattice points of $P \cap \mathbb{Z}^n$. To prove this, observe that if α^* is the map $(\mathbb{C}^*)^r \to (\mathbb{C}^*)^n$ from (3.4) induced by α from (3.2), then

$$t^m \circ \alpha^* = \prod_{i=1}^r x_i^{\langle m, n_i\rangle}$$

for $m \in \mathbb{Z}^n$. Then one computes that

$$x_1^{a_1} \cdots x_r^{a_r}\, t^m \circ \alpha^* = \prod_{i=1}^r x_i^{a_i} \prod_{j=1}^n \prod_{i=1}^r x_i^{\langle m, n_i\rangle} = \prod_{i=1}^r x_i^{\langle m, n_i\rangle + a_i} = x^{[m]}.$$

When restricted to a point in $(\mathbb{C}^*)^r$, it follows that as we vary $m \in P \cap \mathbb{Z}^n$, the functions $t^m \circ \alpha^*$ and $x^{[m]}$ differ by a multiplicative factor which doesn't depend on m. Hence (4.10) and the map φ from (4.4) give the same map on $(\mathbb{C}^*)^n$. In particular, this proves the claim made earlier that φ extends to all of X_P.

4.5. **The Dehn-Sommerville Equations.** Euler's formula for a 3-dimensional polytope $Q \subset \mathbb{R}^3$ states that

$$f_0 - f_1 + f_2 = 2, \tag{4.11}$$

where f_i is the number of i-dimensional faces of Q. If Q has the additional property that all of its facets are triangles (such as a tetrahedron, octahedron or icosahedron), then counting edges gives

$$3f_2 = 2f_1. \tag{4.12}$$

To generalize these, suppose that Q is an n-dimensional polytope in $\mathbb{R}^n$ such that every facet is *simplicial*, meaning that every facet has exactly n vertices. For such a polytope, let f_i be the number of i-dimensional faces of Q, and let $f_{-1} = 1$. Then, for $0 \le p \le n$, set

$$h^p = \sum_{i=p}^n (-1)^{i-p} \binom{i}{p} f_{n-i-1}.$$

The *Dehn-Sommerville equations* assert that if $Q \subset \mathbb{R}^n$ is an n-dimensional simplicial polytope, then

$$h^p = h^{n-p} \quad \text{for all } 0 \le p \le n. \tag{4.13}$$

When $n = 3$, (4.11) is $h^0 = h^3$ and (4.12) is equivalent to $h^1 = h^2$ (assuming $h^0 = h^3$).

To prove (4.13), note that we can move Q so that the origin is an interior point. Furthermore, wiggling the vertices by a small amount does not change the combinatorial type of Q. Thus we may assume that its vertices lie in $\mathbb{Q}^n$. The polar Q° also has vertices in $\mathbb{Q}^n$, so that $P = \nu Q^\circ$ is a lattice polytope for suitably chosen $\nu \in \mathbb{Z}$. Then, as in Section 4.2, the normal fan of P is obtained by projecting from origin to the faces of Q. This fan is simplicial since Q is, so that X_P is a simplicial projective toric variety.

Being projective and simplicial implies two nice facts about X_P:

- $h^p = \dim\, H^{2p}(X_P, \mathbb{Q})$ for $0 \leq p \leq n$.
- Poincaré Duality holds for X_P, i.e., $\dim\, H^q(X_P, \mathbb{Q}) = \dim\, H^{2n-q}(X_P, \mathbb{Q})$ for $0 \leq q \leq 2n$.

The Dehn-Sommerville equations (4.13) follow immediately! In the smooth case, the second bullet is Poincaré Duality. The simplicial case is similar since a quasismooth variety is a rational homology manifold.

This is very pretty but is not the end of the story. One can also ask if it is possible to characterize *all* possible vectors $(f_0, f_1, \ldots, f_{n-1})$ coming from n-dimensional simplicial polytopes. For example, when $n = 3$, one can show that a vector of positive integers (f_0, f_1, f_2) comes from a 3-dimensional simplicial polytope if and only if $f_0 \geq 4$ and (4.11) and (4.12) are satisfied. This can be generalized to arbitrary dimensions, though the result takes some work to state. A nice account can be found in Section 5.6 of [17]. What's interesting is that the proof uses the Hard Lefschetz Theorem for simplicial toric varieties (which is a very difficult theorem).

4.6. **The Ehrhart Polynomial.** In our discussion of the toric variety of a lattice polytope P, we encountered lattice points in positive integer multiples of P (to get a projective embedding) and in the interior of P (Exercise 4.2). The following wonderful result of Ehrhart describes the *number* lattice points in positive integer multiples of a lattice polytope and its interior.

Theorem 4.4. *Let P be an n-dimensional lattice polytope in $M_{\mathbb{R}} = \mathbb{R}^n$. Then there is a unique polynomial E_P (the **Ehrhart polynomial**) with coefficients in $\mathbb{Q}$ which has the following properties:*

(1) *For all integers $\nu \geq 0$,*
$$E_P(\nu) = |(\nu P) \cap M|$$
(2) *If the volume is normalized so that the unit n-cube determined by a basis of M has volume 1, then the leading coefficient of E_P is* $\mathrm{vol}(P)$.
(3) *If* $\mathrm{int}(P)$ *is the interior of P, then the* **reciprocity law** *states that for all integers $\nu > 0$,*
$$E_P(-\nu) = (-1)^n\, |(\nu \mathrm{int}(P)) \cap M|.$$

Before discussing the proof, we will give a classic application in dimension 2. If P is a lattice polygon, then the Ehrhart polynomial is
$$E_P(x) = \mathrm{area}(P)\, x^2 + B\, x + 1 \tag{4.14}$$
since $E_P(0) = |(0 \cdot P) \cap M| = 1$. If we let ∂P denote the boundary of P, then
$$E_P(1) = |P \cap M| = |\mathrm{int}(P) \cap M| + |\partial P \cap M| = E_P(-1) + |\partial P \cap M|,$$
where the last equality uses the reciprocity law. By (4.14), we also have
$$E_P(1) = \mathrm{area}(P) + B + 1 \quad \text{and} \quad E_P(-1) = \mathrm{area}(P) - B + 1.$$
Combining these equalities gives the following:

- $B = \frac{1}{2}|\partial P \cap M|$, so that the Ehrhart polynomial of a lattice polygon is
$$E_P(x) = \mathrm{area}(P)\, x^2 + \tfrac{1}{2}|\partial P \cap M|\, x + 1.$$
- In particular, setting $x = 1$ gives *Pick's Formula*
$$|P \cap M| = \mathrm{area}(P) + \tfrac{1}{2}|\partial P \cap M| + 1.$$

While Theorem 4.4 can be proved by elementary means, my favorite proof uses the cohomology of line bundles on the toric variety X_P. The full details go beyond the scope of these lectures (see [12] for a complete proof), so we will just present the highlights of the proof.

Recall from Section 4.3 that X_P has the ample divisor $D_P = \sum_F a_F D_F$ and that for any positive integer ν, we have $X_P = X_{\nu P}$ and $D_{\nu P} = \nu D_P$. Furthermore, applying Proposition 4.3 to νP gives

$$\dim H^0(X_P, \mathcal{O}_{X_P}(\nu D_P)) = |(\nu P) \cap M|.$$

We define the *Euler-Poincaré characteristic* of $\mathcal{O}_{X_P}(\nu D_P)$ to be

$$\chi(X_P, \mathcal{O}_{X_P}(\nu D_P)) = \sum_{i=0}^{n} (-1)^i \dim H^i(X_P, \mathcal{O}_{X_P}(\nu D_P)).$$

The *Riemann-Roch theorem* implies that there is a polynomial $h(x) \in \mathbb{Q}[x]$ of degree at most n such that

$$\chi(X_P, \mathcal{O}_{X_P}(\nu D_P)) = h(\nu) \tag{4.15}$$

for *all* integers ν.

The divisors νD_P are ample for $\nu > 0$, so that

$$H^i(X_P, \mathcal{O}_{X_P}(\nu D_P)) = 0, \quad i > 0,$$

by the *Demazure vanishing theorem*, and the same vanishing holds when $\nu = 0$. Using this, the Euler-Poincaré characteristic simplifies to

$$\chi(X_P, \mathcal{O}_{X_P}(\nu D_P)) = \dim H^0(X_P, \mathcal{O}_{X_P}(\nu D_P)) = |(\nu P) \cap M|.$$

Combining this with (4.15), we conclude that the polynomial h satisfies

$$|(\nu P) \cap M| = h(\nu) \tag{4.16}$$

for all $\nu \geq 0$. If we set $E_P(x) = h(x)$, then the first assertion of Theorem 4.4 follows.

The second assertion follows easily from the first, for if $E_P(x) = a_n x^n + \cdots + a_0$, then

$$a_n = \lim_{\nu \to \infty} \frac{E_P(\nu)}{\nu^n} = \lim_{\nu \to \infty} \frac{|(\nu P) \cap M|}{\nu^n} = \mathrm{vol}(P).$$

The proof of the third assertion is more sophisticated. The *dualizing sheaf* of X_P is given by

$$\omega_{X_P} = \mathcal{O}_{X_P}(-\textstyle\sum_F D_F).$$

Since X_P is Cohen-Macaulay, *Serre duality* implies that

$$H^i(X_P, \mathcal{O}_{X_P}(-\nu D_P)) \simeq H^{n-i}(X_P, \mathcal{O}_{X_P}(\nu D_P) \otimes \omega_{X_P})^*.$$

In terms of the Euler-Poincaré characteristic, this easily implies

$$\chi(X_P, \mathcal{O}_{X_P}(-\nu D_P)) = (-1)^n \chi(X_P, \mathcal{O}_{X_P}(\nu D_P) \otimes \omega_{X_P})$$

for all ν. If we combine this with (4.15), we see that the Ehrhart polynomial $E_P = h$ satisfies

$$E_P(-\nu) = (-1)^n \chi(X_P, \mathcal{O}_{X_P}(\nu D_P) \otimes \omega_{X_P})$$

for all ν. But νD_P is ample when $\nu > 0$, so that by the *Kodaira vanishing theorem*,

$$H^i(X_P, \mathcal{O}_{X_P}(\nu D_P) \otimes \omega_{X_P}) = 0, \quad \nu > 0.$$

Hence, for these ν, the above formula simplifies to

$$E_P(-\nu) = (-1)^n \dim H^0(X_P, \mathcal{O}_{X_P}(\nu D_P) \otimes \omega_{X_P}).$$

The final step in the proof is to show that

$$\dim H^0(X_P, \mathcal{O}_{X_P}(\nu D_P) \otimes \omega_{X_P}) = |(\nu \mathrm{int}(P) \cap M|$$

when $\nu > 0$. By replacing P with νP, it suffices to prove this for $\nu = 1$. Note that

$$\mathcal{O}_{X_P}(D_P) \otimes \omega_{X_P} = \mathcal{O}_{X_P}\big(D_P - \textstyle\sum_F D_F\big) = \mathcal{O}_{X_P}\big(\textstyle\sum_F (a_F - 1) D_F\big),$$

where $P = \bigcap_F \{m \in M_{\mathbb{R}} \mid \langle m, n_F\rangle \geq -a_F\}$. Since

$$\mathrm{int}(P) \cap M = \bigcap_F \{m \in M \mid \langle m, n_F\rangle > -a_F\} = \bigcap_F \{m \in M \mid \langle m, n_F\rangle \geq -(a_F - 1)\},$$

the methods used to prove Proposition 4.3 imply that

$$H^0\big(X_P, \mathcal{O}_{X_P}(\textstyle\sum_F (a_F - 1) D_F)\big) = \bigoplus_{m \in \mathrm{int}(P) \cap M} \mathbb{C} \cdot \chi^m.$$

This completes the proof of the theorem!

We conclude with another application of the Ehrhart polynomial. Let $\mathcal{A} = \{m_1, \ldots, m_s\} \subset \mathbb{Z}^n$ and assume that $(1, \ldots, 1)$ is in the row space of the $n \times s$ matrix $\mathbf{A}$ whose columns are $\mathcal{A}$. In this situation, we showed in Section 3.4 of LECTURE 3 that the toric ideal $I_{\mathcal{A}}$ is homogeneous and hence defines the (possibly non-normal) projective toric variety $Y_{\mathcal{A}} \subset \mathbb{P}^{s-1}$. We now give a criterion for $Y_{\mathcal{A}}$ to be normal which involves the Ehrhart polynomial of the polytope $P = \mathrm{Conv}(\mathcal{A})$.

To state the criterion, we define the *Hilbert polynomial* of $Y_{\mathcal{A}}$ to be the unique polynomial $H_{\mathcal{A}}$ for which

$$H_{\mathcal{A}}(\nu) = \#\{m_{i_1} + \cdots + m_{i_\nu} \mid m_{i_1}, \ldots, m_{i_\nu} \in \mathcal{A}\}$$

for $\nu \gg 0$. One can show that the polynomials $H_{\mathcal{A}}$ and E_P have the same leading term, which is the normalized volume of P. Then we have the following result of Sturmfels [34, Theorem 13.11].

Theorem 4.5. *The toric variety $Y_{\mathcal{A}} \subset \mathbb{P}^{\ell-1}$ is normal if and only if the Hilbert polynomial $H_{\mathcal{A}}$ equals the Ehrhart polynomial E_P.*

Lecture 5. Toric Regularity

This lecture will discuss some of the commutative algebra related to toric varieties. We will focus mostly but not exclusively on the concept of *regularity*. Our treatement of regularity will assume familiarity with local cohomology. The commutative algebra we need can be found in [13].

5.1. **Regularity.** We begin by reviewing regularity in the classical case. Let $R = \mathbb{C}[x_0, \ldots, x_n]$, where $\deg(x_i) = 1$ for all i, and let $B = \langle x_0, \ldots, x_n \rangle$ be the irrelevant ideal. Note that R is the homogeneous coordinate ring of the toric variety $\mathbb{P}^n$.

There are several ways to define regularity; we will use the definition given in terms of local cohomology because it generalizes best to the toric context. Given a finitely generated graded R-module M, its local cohomology group $H^i_B(M)$ is also a graded R-module. The graded piece in degree m will be denoted $H^i_B(M)_m$.

Definition 5.1. A finitely generated graded R-module M is *m-regular* if

$$H^i_B(M)_\ell = \{0\} \text{ for all } \ell + i \geq m + 1.$$

The ***regularity*** of M, denoted $\operatorname{reg}(M)$, is the least m for which M is m-regular.

One can also compute $\operatorname{reg}(M)$ from a minimal graded free resolution of M. More precisely, given such a resolution, say

$$0 \longrightarrow F_r \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0, \tag{5.1}$$

write $F_i = \bigoplus_j R(-a_{ij})$. In [14] it is shown that

$$\operatorname{reg}(M) = \max_{i,j}\{a_{ij} - i\}. \tag{5.2}$$

Example 5.1. Suppose that $R = \mathbb{C}[x, y, z]$ and $M = R/\langle x, y^3, z^3 \rangle$. We get the minimal resolution

$$0 \longrightarrow R(-7) \longrightarrow R(-4)^2 \oplus R(-6) \longrightarrow R(-1) \oplus R(-3)^2 \longrightarrow R \longrightarrow M \longrightarrow 0$$

(this follows because x, y^3, z^3 form a regular sequence, so that the free resolution of the ideal they generate is given by the Koszul complex.) Thus

$$\operatorname{reg}(M) = \max\{0-0, 1-1, 3-1, 4-2, 6-2, 7-3\} = 4.$$

In Macaulay 2, the `betti` command gives the following output for the above resolution of M:

```
total: 1 3 3 1
    0: 1 1 . .
    1: . . . .
    2: . 2 2 .
    3: . . . .
    4: . . 1 1
```

In this diagram, an entry b_{ij} in column i and row j indicates that F_i has $R(-i-j)^{b_{ij}}$ as a direct summand, where the rows and columns are numbered starting at 0. Thus the first 2 in the middle of the diagram is in column 1 and row 2, meaning that $R(-1-2)^2 = R(-3)^2$ appears in F_1. The second 2 is in column 2 and row 2, so that $R(-2-2)^2 = R(-4)^2$ appears in F_2. □

In general, the number before the colon on the bottom row of a betti diagram (4 in the above case) is the regularity of M. Be sure you understand this. It follows that the regularity gives a good measure of the computational complexity of M.

Regularity can also be formulated in terms of Ext, as is done in [13]. Local duality provides the link between this definition and the one given above in terms of local cohomology.

Here is one case where regularity is easy to understand.

Example 5.2. Let $I \subset R$ be a homogeneous ideal such that $\mathbf{V}(I) \subset \mathbb{P}^n$ is empty. Then the graded quotient R/I is a finite dimensional vector space over $\mathbb{C}$. In this situation, one can show that

- $H^0_B(R/I) = R/I$.
- $H^i_B(R/I) = \{0\}$ for $i > 0$.

It follows that $\operatorname{reg}(R/I) = d$, where d is the largest integer such that $(R/I)_d \neq 0$, i.e., $I_d \neq R_d$. Thus $\operatorname{reg}(R/I)$ is the smallest integer d such that $I_\ell = R_\ell$ for all $\ell > d$. □

Given a finitely generated graded R-module M, its *Hilbert function* H_M is defined by $H_M(d) = \dim M_d$. A basic result of commutative algebra states that $H_M(d)$ is a polynomial for $d \gg 0$. This is the *Hilbert polynomial* P_M. The regularity of M tells us how large d needs to be in order for these to agree. The precise result is that

$$H_M(d) = P_M(d)$$

for all $d \geq \operatorname{reg}(M) + r - n$, where r is from the minimal free resolution (5.1) and n is from $R = \mathbb{C}[x_0, \ldots, x_n]$. This is proved in [14, Thm. 4.2].

The general intuition is that the regularity captures the computational complexity of a graded R-module. The state-of-the-art for computing regularity is discussed in [2]. See also [28] for another introduction to regularity.

5.2. **Generic Initial Ideals.** A homogeneous ideal $I \subset R$ has an initial ideal $\operatorname{in}(I)$ with respect to graded reverse lexicographic order. However, if we make a generic change of coodinates and then take the initial ideal, we get the *generic initial ideal* $\operatorname{Gin}(I)$. See [13, 15.9] for a discussion of generic initial ideals.

A key result of Bayer and Stillman is that

$$\operatorname{reg}(R/I) = \operatorname{reg}(R/\operatorname{Gin}(I)). \tag{5.3}$$

Furthermore, the regularity of I is the highest degree of a minimal generator of $\operatorname{Gin}(I)$.

A monomial ideal $J \subset R$ is *strongly stable* if $x_i m \in J$ implies $x_j m \in I$ for any $1 \leq j \leq i$. One can prove that generic initial ideals are strongly stable. There is also the related concept of *Borel fixed* ideal, which we won't discuss here (see [13]).

5.3. **Weighted Regularity.** We next discuss regularity in the case of a weighted polynomial ring $R = \mathbb{C}[x_0, \ldots, x_n]$, where $\deg(x_i) = q_i$ for posiive integers $q_0, \ldots, q_n$ satisfying $\gcd(q_0, \ldots, q_n) = 1$. We will also assume that $q_0 \leq \cdots \leq q_n$. The irrelevant ideal $B = \langle x_0, \ldots, x_n \rangle$ is the same as before. Here, R is the homogeneous coordinate ring of the toric variety $\mathbb{P}(q_0, \ldots, q_n)$.

The recent preprint [11] studies regularity in this weighted situation. Given a finitely generated graded R-module M, we define its *weighted regularity* $\operatorname{reg}(M)$ exactly as in Definition 5.1.

Although the definition looks the same, weighted regularity differs from the ordinary notion in several ways. For example, given a minimal graded free resolution (5.1) with $F_i = \bigoplus_j R(-a_{ij})$, [11] proves that

$$\operatorname{reg}(M) = \max_{i,j}\{a_{ij} - i\} - \textstyle\sum_{k=0}^n (q_k - 1). \tag{5.4}$$

Comparing this to (5.2), we see that we need to take the weights of the variables into account when thinking about regularity in terms of a minimal resolution. Note also that (5.4) reduces to (5.2) when the weights are all 1.

5.4. **Weighted Generic Initial Ideals.** The paper [11] explains how to define the generic initial ideal $\mathrm{Gin}(I)$ of a weighted homogeneous ideal $I \subset R$ with respect to weighted reverse lexicographic order. However, one needs to make some assumptions on the weights in order for this to be useful. If $q_i | q_{i+1}$ for $i = 0, \dots, n-1$, then (5.3) continues to hold for weighted regularity.

One way to understand this is to observe that coordinate changes used in the original definition of $\mathrm{Gin}(I)$ involve the automorphism group of $\mathbb{P}^n$. In the weighted case, $\mathbb{P}(q_0, \dots, q_n)$ may have a much smaller automorphism group. For example, when the weights are $q_0 = n+2, q_2 = n+3, \dots, q_n = 2(n+1)$, the corresponding automorphism group is as small as possible, namely $(\mathbb{C}^*)^n$. On the other hand, the divisibility condition $q_i | q_{i+1}$ guarantees a rich supply of automorphisms, hence a better theory of generic initial ideals.

Another way to see the impact of the weights is to consider strongly stable ideals in the weighted case. Here, a monomial ideal $J \subset R$ is *strongly stable* if $x_i m \in J$ implies that $um \in J$ for any monomial $u \in \mathbb{C}[x_1, \dots, x_{i-1}]$ of degree q_i. One can show that in the weighted case, generic initial ideals are strongly stable. However, this might not say much. For instance, Example 1.6 of [11] shows that *every* monomial ideal is strongly stable for the weights $q_0 = n+2, q_2 = n+3, \dots, q_n = 2(n+1)$ mentioned above.

5.5. **Divisors on a Toric Variety.** We have now defined regularity for the homogeneous coordinate rings of $\mathbb{P}^n$ and $\mathbb{P}(q_0, \dots, q_n)$. Before we can extend the definition to the coordinate ring of an arbitrary projective toric variety, we need to say more about divisors on toric varieties.

Let $X = X_\Sigma$ be a projective toric variety with quotient representation $X = (\mathbb{C}^{\Sigma(1)} \setminus \mathbf{V}(B))/G$ as explained in LECTURE 3. This has the homogeneous coordinate ring $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$, which is graded by the Chow group $A_{n-1}(X)$ via the exact sequence (3.2) from LECTURE 3.

Given a Cartier torus-invariant divisor $D = \sum_\rho a_\rho D_\rho$, we follow (4.3) from LECTURE 4 and set

$$H^0(X_P, \mathcal{O}_{X_P}(D)) = \{f \in \mathbb{C}(X_P)^* \mid \mathrm{div}(f) + D \geq 0\}.$$

This vector space is spanned by the characters χ^m satisfying $\mathrm{div}(\chi^m) + D \geq 0$. Then we say that D is *generated by global sections* if these characters give an everywhere defined map from X to projective space as described in LECTURE 4. Furthermore, D is *ample* if this map is an embedding for some positive integer multiple of D.

Then define

$$\mathcal{K} = \{[D] \in A_{n-1}(X) \mid D \text{ is Cartier and generated by global sections}\}$$

and its saturation

$$\mathcal{K}^{\mathrm{sat}} = \{[D] \in A_{n-1}(X) \mid \mu D \in \mathcal{K} \text{ for some } \mu > 0\}.$$

One can show that $\mathcal{K}$ and $\mathcal{K}^{\mathrm{sat}}$ are finitely generated subsemigroups of the finitely generated Abelian group $A_{n-1}(X)$. Also, $\mathcal{K} = \mathcal{K}^{\mathrm{sat}}$ whenever X is smooth.

Here are some examples.

Example 5.3. Both $\mathbb{P}^n$ and $\mathbb{P}(q_0, \dots, q_n)$ have $\mathbb{Z}$ as their Chow group. For $\mathbb{P}^n$, we have $\mathcal{K} = \mathcal{K}^{\mathrm{sat}} = \mathbb{N}$, while for $\mathbb{P}(q_0, \dots, q_n)$, we have $\mathcal{K}^{\mathrm{sat}} = \mathbb{N}$ and $\mathcal{K} = \mathrm{lcm}(q_0, \dots, q_n)\mathbb{N}$. The latter reflects the fact that a Weil divisor on $\mathbb{P}(q_0, \dots, q_n)$ is Cartier if and only if its class in the Chow group is a multiple of all of the weights. □

Example 5.4. For $\mathbb{P}^1 \times \mathbb{P}^1$, Example 3.1 from LECTURE 3 shows that the Chow group is $\mathbb{Z}^2$. Here, one can prove that $\mathcal{K} = \mathcal{K}^{\mathrm{sat}} = \mathbb{N}^2$. □

There is a *lot* more to say about divisors on toric varieties. The interested reader should consult [17, Ch. 3] and [26, Ch. 2].

5.6. **Toric Regularity.** Let X be a projective toric variety with homogeneous coordinate ring S graded by $A_{n-1}(X)$ and irrelevant ideal $B \subset S$. In [24], Maclagan and Smith define the regularity of a finitely generated S-module M. Their definition involves picking a finite set $\mathcal{C} \subset A_{n-1}(X)$. For simplicity, we will assume that $\mathcal{C}$ consists of the minimal generators of $\mathcal{K}^{\text{sat}} \subset A_{n-1}(X)$. This implies that $\mathbb{N}\mathcal{C} = \mathcal{K}^{\text{sat}}$.

Definition 5.2. Let M be a graded S-module and fix $\mathcal{C} = \{c_1, \ldots, c_\ell]\}$ as above. Given $m \in A_{n-1}(X)$, we say that M is m***-regular*** if

$$H^i_B(M)_p = \{0\}$$

for all of the following p:

(1) $i \geq 0$ and p is of the form $p = m - \lambda_1 c_1 - \cdots - \lambda_\ell c_\ell + u$, where $\lambda_1, \ldots, \lambda_\ell \in \mathbb{N}$ satisfy $\lambda_1 + \cdots + \lambda_\ell = i - 1$ and $u \in \mathcal{K}^{\text{sat}}$.
(2) $i = 0$ and p is of the form $p = m + c_j + u$, where $1 \leq j \leq \ell$ and $u \in \mathcal{K}^{\text{sat}}$.

Here is an example.

Example 5.5. For $X = \mathbb{P}^n$, we have $A_{n-1}(X) = \mathbb{Z}$, $\mathcal{K}^{\text{sat}} = \mathbb{N}$, and $\mathcal{C} = \{c_1\} = \{1\}$. Thus M is m-regular provided $H^i_B(M)_p = \{0\}$ for $p = m - (i-1) + u$ for $u \in \mathbb{N}$, i.e., $p + i \geq m + 1$. This is exactly the condition appearing in Definition 5.1. The same is true in the weighted case. □

To understand Definition 5.2, the key point is that we need more and more of the local cohomology group $H^i_B(M)$ to vanish as i increases. In the classical case considered in Example 5.5, the vanishing condition becomes $p \geq m + 1 - i$, i.e., each time you increase i by one, you need decrease the lower bound by 1. But in the general case, this lower bound lies in $A_{n-1}(X)$, so that the "decrease" occurs relative to $\mathcal{K}^{\text{sat}} = \mathbb{N}\mathcal{C}$. This is why the $1 - i = -(i-1)$ for $\mathbb{P}^n$ turns into $-\lambda_1 c_1 - \cdots - \lambda_\ell c_\ell$, $\lambda_1 + \cdots + \lambda_\ell = i - 1$, for the toric variety X.

Definition 5.3. Given a graded S-module as above, we define $\text{reg}(M) \subset A_{n-1}(X)$ to be the set

$$\text{reg}(M) = \{m \in A_{n-1}(X) \mid m \text{ is } m\text{-regular}\}.$$

Note that this is a set, not a single element. For toric varieties like $\mathbb{P}^n$ or $\mathbb{P}(q_0, \ldots, q_n)$ that have $\mathbb{Z}$ as their Chow group, one can take the minimal element of $\text{reg}(M)$, which is just the regularity defined earlier.

Here are some examples when $M = S$.

Example 5.6. In Example 3.3 of Lecture 3, we saw that the homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ is $\mathbb{C}[x_1, x_2, x_3, x_4]$, where $\deg(x_1) = \deg(x_2) = (1, 0)$ and $\deg(x_3) = \deg(x_4) = (0, 1)$. To simplify notation, we will write this ring as $S = \mathbb{C}[x, y; z, w]$, where the semicolon reminds us that x, y have degree $(1, 0)$ and z, w have degree $(0, 1)$. Recall from Example 5.4 that $\mathcal{K}^{\text{sat}} = \mathbb{N}^2$, so that $\mathcal{C} = \{e_1, e_2\}$. In Example 4.3 of [24] it is shown that $\text{reg}(S) = \mathbb{N}^2$. This can also be computed directly from the definition by relating $H^i_B(S)$ to the sheaf cohomology groups

$$H^i(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b))$$

and using known vanishing theorems.

Example 5.7. When S is the homogeneous coordinate ring of the weighted projective space $\mathbb{P}(q_0, \ldots, q_n)$, we have the free resolution $0 \to S \to S \to 0$. By the formula for weighted regularity given in (5.4), we obtain

$$\text{reg}(S) = \max\{0 - 0\} - \textstyle\sum_{k=0}^n (q_k - 1) = n + 1 - \sum_{k=0}^n q_k,$$

where we are thinking of regularity as a number. In contrast, Example 4.2 of [24] uses topological methods to compute $H^i_B(S)$, with the result that

$$\operatorname{reg}(S) = \{m \in \mathbb{Z} \mid m \geq n + 1 - \textstyle\sum_{k=0}^n q_k\},$$

now thinking of regularity as a set. These are clearly consistent. □

These examples have $0 \in \operatorname{reg}(S)$. However, Example 6.11 of [24] shows that this is not always the case. It is not known if the class of an ample divisor always lies in $\operatorname{reg}(S)$. Another question concerns the theory of generic initial ideals. Since many toric varieties have only the torus as automorphism group (see Section 5.4 or Example 4.11 of [24]), the problem is to characterize those toric varieties X for which $\operatorname{Aut}(X)$ is large enough to give a good theory of generic initial ideals. The case of $\mathbb{P}^n \times \mathbb{P}^m$ is discussed in [1].

5.7. **Minimal Generators.** In the case of $\mathbb{P}^n$, the regularity is a number, and (5.2) shows that $\operatorname{reg}(M)$ is an upper bound for the degrees of the minimal generators of M. In the toric case, Theorem 5.4 of [24] explains how minimal generators relate to regularity.

Theorem 5.4. *If X is smooth, then the degrees of the minimal generators of M lie outside the set*

$$\operatorname{reg}(M) + \bigcup_{j=1}^{\ell} (c_j + \mathcal{K}^{\mathrm{sat}}).$$

The following example shows to what extent the regularity restricts the degrees of the minimal generators.

Example 5.8. For $\mathbb{P}^1 \times \mathbb{P}^1$ and $S = \mathbb{C}[x, y; z, w]$, we will consider the ideal $I = \langle xy, zw\rangle \subset S$. According to Example 1.1 of [30], we have

$$\operatorname{reg}(S/I) = (1, 1) + \mathbb{N}^2.$$

It follows from Theorem 5.4 that the minimal generators of S/I do not lie in

$$(1, 1) + \mathbb{N}^2 + \big((e_1 + \mathbb{N}^2) \cup (e_2 + \mathbb{N}^2)\big) = ((2, 1) + \mathbb{N}^2) \cup ((1, 2) + \mathbb{N}^2).$$

Once we exclude this set, we still have infinitely many elements of $\mathbb{N}^2$ to choose from. So the regularity does not bound the degrees of the minimal generators in this case. □

In order to avoid this problem, the paper [30] uses "coarsenings" of the grading on S, which makes S into a $\mathbb{Z}$-graded algebra. This allows one to get bounds on degrees of the minimal generators as well as bounds on the degrees of the higher syzygies.

5.8. **Regularity and Resolutions.** In Example 5.7, we were able to use a free resolution to compute the regularity. Unfortunately, the relation between $\operatorname{reg}(M)$ and a free resolution of M is more complicated for an arbitrary projective toric variety.

To state the result, we need some notation and terminology. First, an finitely generated S-module M is *B-torsion* if $B^k M = \{0\}$ for $k \gg 0$. Then consider

$$0 \longrightarrow F_s \xrightarrow{\partial_s} \cdots \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \longrightarrow 0. \tag{5.5}$$

We say that (5.5) is a *B-torsion resolution* of M if:

(1) Each F_i is a free graded S-module.
(2) (5.5) is a complex, i.e., $\partial_i \circ \partial_{i+1} = 0$ for all i.
(3) $\ker(\partial_i)/\operatorname{im}(\partial_{i+1})$ is B-torison.
(4) ∂_0 is surjective.

Given such a resolution, write

$$F_i = \bigoplus_{j=1}^{\ell_i} S(-a_{ij}), \quad a_{ij} \in A_{n-1}(X).$$

With this set-up, we have Theorem 1.5 of [24]:

Theorem 5.5. *If $\phi : \{2, \dots, \min\{n+1, s\}\} \to \{1, \dots, \ell\}$ is any function, then the intersection*

$$\bigcap_{j=1}^{\ell_0} \big(a_{0j} + \operatorname{reg}(S)\big) \cap \bigcap_{k=1}^{\ell} \bigcap_{j=1}^{\ell_1} \big(a_{1j} - e_k + \operatorname{reg}(S)\big) \cap \bigcap_{i=2}^{\min\{n+1,s\}} \bigcap_{k=1}^{\ell} \bigcap_{j=1}^{\ell_i} \big(a_{ij} - e_k - e_{\phi(2)} - \cdots - e_{\phi(i)} + \operatorname{reg}(S)\big)$$

is contained in $\operatorname{reg}(M)$.

This differs slightly from the result stated in [24]. I am grateful to the authors of [24] for sending me the corrected version of the theorem.

Here is an example of how to use Theorem 5.5.

Example 5.9. Consider the ideal $I = \langle xy, zw \rangle \subset S = \mathbb{C}[x, y; z, w]$ from Example 5.8. Since xy, zw are relatively prime, they form regular sequence. Hence their Koszul complex gives a resolution

$$0 \longrightarrow S(-2,-2) \longrightarrow S(-2,0) \oplus S(0,-2) \longrightarrow S \longrightarrow S/I \longrightarrow 0. \tag{5.6}$$

If $\phi : \{2\} \to \{1, 2\}$ is any map, then Theorem 5.5 implies that

$$\begin{aligned}&\mathbb{N}^2 \cap ((2,0) - e_1 + \mathbb{N}^2) \cap ((2,0) - e_2 + \mathbb{N}^2) \cap ((0,2) - e_1 + \mathbb{N}^2) \cap ((0,2) - e_2 + \mathbb{N}^2) \cap \\ &((2,2) - e_1 - e_{\phi(2)} + \mathbb{N}^2) \cap ((2,2) - e_2 - e_{\phi(2)} + \mathbb{N}^2) = (2,2) + \mathbb{N}^2 \subset \operatorname{reg}(S/I).\end{aligned}$$

However, we saw in Example 5.8 that $\operatorname{reg}(S/I) = (1,1) + \mathbb{N}^2$. Hence the subset $(2,2) + \mathbb{N}^2$ coming from (5.6) via Theorem 5.5 is a proper subset of $\operatorname{reg}(S/I)$. This shows that a free resolution does not determine the regularity, unlike what happens in the case of $\mathbb{P}^n$ or $\mathbb{P}(q_0, \dots, q_n)$. □

We should mention that [24] discusses regularity for a more general class of multigraded rings that includes the homogeneous coordinate ring of a toric variety as special cases. A good reference for multigraded rings is [25].

5.9. **Bigraded Regularity.** We now concentrate on the bigraded ring $S = \mathbb{C}[x, y; z, w]$ that has appeared in numerous examples in this lecture. The notion of bigraded regularity was first defined in [22], where it was called *weak regularity*. This concept is close but not identical to what we get by applying Definition 5.2 to $\mathbb{P}^1 \times \mathbb{P}^1$. The authors of [22] also define *strong regularity*, which they are able to compute in terms of a free resolution.

In the bigraded case, there is also *regularity vector*, first considered by Aramova, Crona and De Negri [1], then studied further in [27] and generalized in [31]. This can be defined from a minimal bigraded free resolution

$$0 \longrightarrow F_s \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

as follows. Write $F_i = \oplus_j S(-a_{ij}, -b_{ij})$, where $(-a_{ij}, -b_{ij}) \in \mathbb{Z}^2$. Then the *resolution regularity vector* (this is the terminology of [31]) is

$$\underline{r} = (r_1, r_2),$$

where

$$r_1 = \max_{ij}\{a_{ij} - i\}, \quad r_2 = \max_{ij}\{b_{ij} - i\}.$$

Results about $\underline{r}$ and its relation to Definition 5.2 are discussed in [19, 31].

5.10. **Bigraded Commutative Algebra.** As we've seen, the concept of regularity is complicated in the bigraded case. We will end this lecture with an example, taken from [8], that illustrates some of the further difficulties of doing bigraded commutative algebra.

To understand the problems that will arise, we first consider what happens in $\mathbb{P}^2$. Suppose that f_0, f_1, f_2 are homogeneous of degree 3 in $R = \mathbb{C}[x, y, z]$ $(\deg(x) = \deg(y) = \deg(z) = 1)$ that don't vanish simultaneously on $\mathbb{P}^2$. Then one can prove without difficulty that

- The Koszul complex of f_0, f_1, f_2 is exact, and
- The polynomials f_0, f_1, f_2 form a regular sequence.

The first bullet shows that the minimal free resolution of $I = \langle f_0, f_1, f_2\rangle \subset R$ is given by

$$0 \longrightarrow R(-9) \longrightarrow R(-6)^3 \longrightarrow R(-3)^3 \longrightarrow I \longrightarrow 0.$$

It follows that the shape of the minimal free resolution is determined by the geometric assumption that f_0, f_1, f_2 have no common zeros in $\mathbb{P}^2$.

Now switch to $\mathbb{P}^1 \times \mathbb{P}^1$ and let f_0, f_1, f_2 be homogeneous polynomials in $S = \mathbb{C}[x, y; z, w]$ of degree $(2, 1)$ that don't vanish simultaneously on $\mathbb{P}^1 \times \mathbb{P}^1$. In this case, one can prove (see [8]) that

- The Koszul complex of f_0, f_1, f_2 is *not* exact, and
- The polynomials f_0, f_1, f_2 *do not* form a regular sequence.

Furthermore, when vary over all triples f_0, f_1, f_2 satisfying our two conditions (degree $(2, 1)$ and no common zeros on $\mathbb{P}^1 \times \mathbb{P}^1$), we first that there are *two* possible shapes for the minimal free resolution of $I = \langle f_0, f_1, f_2\rangle \subset S$.

To state the precise result, let $S_{2,1}$ be the graded piece of S in degree $(2, 1)$ and let $Y \subset S_{2,1}$ be the subvariety determined by the image of multiplication map $S_{2,0} \times S_{0,1} \to S_{2,1}$. We also have the 3-dimensional subspace $W = \mathrm{Span}(f_0, f_1, f_2) \subset S_{2,1}$. One of the main results of [8] is that if $W \cap Y$ has dimension 2 (which happens generically), then the minimial free resolution of I is

$$0 \to R(-6,-3) \to \begin{matrix} R(-4,-3)^3 \\ \oplus \\ R(-6,-2)^2 \end{matrix} \to \begin{matrix} R(-6,-1) \\ \oplus \\ R(-4,-2)^3 \\ \oplus \\ R(-3,-3)^2 \end{matrix} \to R(-2,-1)^3 \to I \to 0,$$

while if $W \cap Y$ has dimension 1 (the only other possibility given our hypotheses), then the minimial free resolution is

$$0 \to R(-6,-3) \to \begin{matrix} R(-4,-3)^2 \\ \oplus \\ R(-6,-2)^2 \end{matrix} \to \begin{matrix} R(-6,-1) \\ \oplus \\ R(-4,-2)^3 \\ \oplus \\ R(-2,-3) \end{matrix} \to R(-2,-1)^3 \to I \to 0.$$

Hence the nice relation between the geometry (no common zeros) and the algebra (the shape of the free resolution) is more complicated in the bigraded case.

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